Spectral problems in the shape optimisation, singular boundary perturbations

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SPECTRAL PROBLEMS IN THE SHAPE OPTIMISATION. SINGULAR BOUNDARY PERTURBATIONS

S.A. NAZAROV AND J. SOKOLOWSKI

Abstract. In the paper asymptotic analysis of spectral problems is performed for singular perturbations of geometrical domains. Asymptotic approximations of eignevalues and eigenfunctions are constructed for the scalar, second order boundary value problems. The presented results are constructive and can be used in the analysis of shape optimization and inverse problems.

Keywords: Singular perturbations; Spectral problem; Asymptotics of eigenfunctions and eignevalues;
MSC: Primary 35B40, 35C20; Secondary 49Q10, 74P15

§1. Introduction.

1.1. **Shape optimisation problems for eigenvalues.** In the paper asymptotic analysis of eigenvalues and eigenfunctions is performed with respect to singular perturbations of geometrical domains (see Fig. 2). The case of low frequencies is considered for scalar equations in two spatial dimensions. Similar results for elastic bodies are presented in a forthcoming paper. The results established here can be directly used in some applications, for example in shape sensitivity analysis of the Helmholtz equation. Compared to the existing results in the literature, the technical difficulties of the present paper concern the variable coefficients of differential operators in limit problems that particularly arise from the curved boundaries. The known results are given for singular perturbations of isolated points of the boundary (small holes in the domain, see [26], [27], [16], [5], [28], [39]-[40]), perturbations of straight boundaries including perturbations by changing the type of boundary conditions (cf. [7]-[10]), and the dependence of the obtained results in more general geometrical domains on the curvature is not clarified up to now. We show that the first order correction terms are independent on the curvature, even if the appropriate change of curvilinear variables leads to differential expressions depending explicitly on the curvature. The perturbations of boundaries in the form of caverns, so we take off some material, and knops or proturberances, so we add some material, cannot be analysed with the classical tools of shape optimisation. The asymptotic analysis seems to be the only avalaible tool to perform the efficient analysis of boundary value problems, eigenvalues and eigenfunctions, and of shape functionals, in general setting. The internal perturbations of the domain by creation of small openings or holes, but very close to the boundary (see Fig. 1), are included into the scheme of asymptotic analysis presented in the paper. In relation to shape optimisation, such an analysis leads to the asymptotic approximations of shape functionals. The first term in such approximations in the specific case of topology changes is called *topological derivative* and can be used in numerical methods e.g., of the level set type. In the case of boundary perturbations, the first term in asymptotic approximations of shape

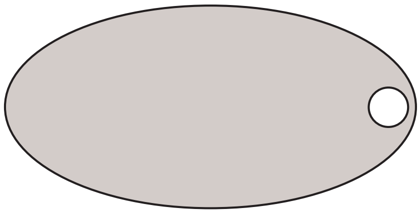

FIGURE 1. Small hole very close to the boundary.

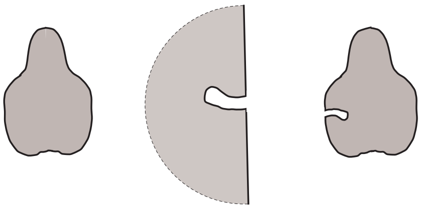

FIGURE 2. The domains Ω, Ξ, and $\Omega(h)$.

functionals replaces the so-called shape gradients which are obtained under much more restrictive assumptions compared to the present paper.

The description of shape optimisation problems for eigenvalues can be found in monographs [44], [3], [13], [12], [2], [48]. There is a natural gap between the regularity of geometrical domains, from one side for the results on the existence of optimal domains, where some weak conditions e.g., in the Dirichlet case of the type Mosco convergence for minimising sequences of admissible domains are required, and the necessary optimality conditions where stronger assumptions on the regularity of boundaries of admissible domains are necessary if the boundary variations technique [44], is applied to compute the directional derivatives of eigenvalues with respect to domain perturbations, even in non-smooth situations of the cracks [6], [23]. The authors filled partially the gap in the paper on topological derivatives [38], in the present paper the non-smooth boundary variations are considered for the particular class of shape functionals.

1.2. **Problem formulation.** Let $\Omega \subset \mathbb{R}^2$ be a domain with the smooth boundary Γ, the boundary is a simple, regular, and closed contour. In the neighbourhood of Γ a curvilinear system of coordinates (n, s) is defined, where s is the length of the curve measured along Γ, n denotes the oriented distance to Γ, while $n > 0$ in $\Omega^c = \mathbb{R}^2 \setminus \overline{\Omega}$. By $\omega \subset \mathbb{R}^2_- = (-\infty, 0) \times \mathbb{R}$ is denoted a domain with the compact closure $\overline{\omega} = \omega \cup \partial\omega$. The boundary $\partial\Xi$ of the infinite domain $\Xi = \mathbb{R}^2_- \setminus \overline{\omega}$ is assumed to be piecewise-smooth, which means that there is a finite set of points $P_1, \ldots, P_N$ on $\partial\Xi$, such that each curvilinear interval P_iP_{i+1} is smooth and the angles between tangents at P_i, $i = 1, \ldots, N$ are strictly positive. In other words, peaks directed outside are forbidden.

Introduce a family of domains, depending on the small parameter $h > 0$,

$$\omega_h = \{x = (x_1, x_2) : \xi = (\xi_1, \xi_2) := (h^{-1}n, h^{-1}s) \in \omega\}, \tag{1.1}$$

$$\Omega(h) = \Omega \setminus \overline{\omega_h} \tag{1.2}$$

(see Fig. 1.2). Here and in the sequel, a point on the contour Γ is identified with its coordinate s, with the convention that for the points which are located on Γ on the left-hand side of the origin O are given the negative values of the parameter s.

Let us consider the spectral Neumann problem

$$-\Delta_x u^h(x) = \lambda^h u^h(x), \quad x \in \Omega(h), \tag{1.3}$$

$$\partial_{n^h} u^h(x) = 0, \quad x \in \Gamma(h) := \partial\Omega(h), \tag{1.4}$$

for the Laplace operator Δ_x, where $\partial_{n^h} = n^h \cdot \nabla_x$ is the normal derivative along the outer normal n^h. Note that (1.3) is but the Helmholtz equation. Conditions (1.4) are prescribed along $\Gamma(h)$ except of the points $P_i(h)$, $i = 1, \ldots, N$, which are images of the points P_i, $i = 1, \ldots, N$, on the contour $\Gamma(h)$. Problem (1.3), (1.4) admits the sequence of eigenvalues

$$0 = \lambda_0^h < \lambda_1^h \le \lambda_2^h \le \cdots \le \lambda_m^h \le \cdots \to +\infty \tag{1.5}$$

where the multiplicity is explicitly indicated. The corresponding eigenfunctions $u_0^h, u_1^h, u_2^h, \ldots, u_m^h, \ldots$ can be subject to the orthogonality and normalisation conditions

$$(u_p^h, u_m^h)_{\Omega(h)} = \delta_{p,m}, \quad p, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}, \tag{1.6}$$

where $(\cdot,\cdot)_\Upsilon$ is the scalar product in the Lebesgue space $L_2(\Upsilon)$, and $\delta_{p,m}$ the Kronecker symbol.

Our aim is the derivation of asymptotic formulae for the eigenvalues and eigenfunctions of problem (1.3), (1.4). It is not difficult to see (cf. §3), that for a fixed index m and with $h \to 0$ the entry λ_m^h of the sequence (1.5) converges to the appropriate element of the sequence

$$0 = \lambda_0^0 < \lambda_1^0 \le \lambda_2^0 \le \cdots \le \lambda_m^0 \le \cdots \to +\infty \tag{1.7}$$

of the eigenvalues for the limit, with $h = 0$, spectral Neumann problem

$$-\Delta_x v^0(x) = \lambda^0 v(x), \quad x \in \Omega; \qquad \partial_n v^0(x) = 0, \quad x \in \Gamma. \tag{1.8}$$

The eigenfunctions of (1.8) are smooth functions in $\overline{\Omega}$ and are subject to the orthogonality and normalisation conditions

$$(v_p^0, v_m^0)_\Omega = \delta_{p,m}, \quad p, m \in \mathbb{N}. \tag{1.9}$$

1.3. **Asymptotic ansätze and procedures.** In the paper the following ansätze are used to construct the asymptotics of eigenvalues and eigenfunctions

$$\lambda_m^h = \lambda_m^0 + h^2 \lambda_m' + \ldots, \tag{1.10}$$

$$u_m^h(x) = v_m^0(x) + h\chi(x) w_m^1(\xi) + h^2 \chi(x) w_m^2(\xi) + h^2 v_m^2(x) + \ldots \tag{1.11}$$

Here v_m^0 and v_m^2 are terms of regular type, a smooth and a continuous function, respectively on the set $\overline{\Omega}$, and w_m^1, w_m^2 are terms of the boundary layer type, which depend on the *rapid variables* $\xi = (\xi_1, \xi_2)$, defined in (1.1), and are given by the solutions of the Neumann problem in the domain Ξ. Finally, $\chi \in C^\infty(\overline{\Omega})$ is the cut-off function, equals to zero outside of a neighbourhood of the origin O and equals to one in the vicinity of the point O.

The procedures of constructions of asymptotic ansätze of the type (1.10), (1.11) as well as the determination of theirs terms are not of an particular interest. During the years 70-80 of the last century the subject was fully investigated in the framework of two methods, of matched [14] and compound [28] asymptotic expansions for examining solutions in domains with singularly perturbed boundaries. In addition, if the domain Ω is included in the half-space bounded by the tangent L to the contour Γ at the point O, and Γ coincides with L in the vicinity of O, then by means of even extension over the part of the boundary (odd for the Dirichlet boundary conditions) we obtain from (1.3), (1.4) the spectral Neumann problem in the domain with small hole (see Fig. 2). For such a problem the asymptotics are obtained in [27] (see also [26], [5], [32], [5], [7], [8], etc. devoted to such class of problems).

For the above reasons, we pay a particular attention in the paper to the dependence of the terms in ansätze (1.10) and (1.11) upon the curvature $\varkappa(O)$ of the contour Γ at the point O. From the dimension analysis it follows directly that curvature is absent in the principal correction terms in the asymptotics of the eigenfunctions and of the eigenvalues. In addition, the principal term of the boundary layer type is also independent of the curvature $\varkappa(O)$ at the point O, however, the terms v_m^2, w_m^2 and λ_m' can be dependent on $\varkappa(O)$. Actually, it is the case for the term w_m^2 but we find out finally that the term λ_m' is independent of $\varkappa(O)$. The proof of this fact is complex, includes some technicalities and it is one of the main results of the paper. We provide the proof in §2, which contains the derivation of terms in representations (1.10) and (1.11).

The structure of ansatz (1.11) shows that in the sequel the method of compound asymptotics expansions is applied. In particular, it is explained in section 2.2, that the only function w^1 enjoys the canonic property of the boundary layer, i.e., it decays for $|\xi| \to \infty$, in contrast to w^2 which has the logarithmic growth at infinity. In this way, decomposition of the terms in ansatz (1.11) into the *regular* and *boundary layer* parts is relatively formal. By an application of the procedure of rearrangement of discrepancies [25] (see also monograph [28]) it is possible to reformulate the ansatz in such a way that the function w^2 becomes decreasing, however in such a case the logarithmic growth passes to the term v^2 which thus looses the regularity. It is convenient for further purposes accept that v^2 is bounded and w^2 enjoys the growth, which particularly simplifies the chosen way of the justification of asymptotics in §3.

1.4. **The structure of the paper.** We briefly describe the contain of paper. In §2 the terms in asymptotic ansätze (1.10), (1.11) are subsequently constructed for the Neumann problem. The explicit formulae for the variations of eigenvalues is given, both in the cases of simple and multiple. In §3 the formal asymptotics is justified. In §4 different boundary conditions are considered including Dirichlet and mixed boundary conditions. The variations of the boundary include perturbations of angular (corner) points as well as smoothing of such points of the boundary. In §5 the associated shape optimisation problems are investigated, using the asymptotic formula already derived, and the asymptotics of shape functionals are constructed.

We point out, that the authors attempt to express the perturbations and the appearing integral characteristics of geometrical objects by means of classical characteristics which include the tensor of virtual mass and polarisation tensor, logarithmic capacity etc. Many exact formulae for classical characteristics of a broad class of canonical shapes are provided in monographs [41], [22] and others.

1.5. **Revisiting shape optimisation.** Our results can be used for shape optimisation of spectral problems, in particular for solutions of the Helmholtz equation. We provide the analysis of non-smooth perturbations of boundaries which uses the same tools as the derivation of topological derivatives of shape functionals, but in the case of domain boundaries. In particular boundary cracks are allowed for the *boundary variations*. In this way we extend the notion of shape gradient to the case of singular boundary perturbations. The immediate conclusion from the obtained formulae can be employed to obtain an information about the decreasing and increasing of eigenvalues for the specific boundary perturbations in the form of caverns and knops. Such an information is interesting on its own for the analysis of shape optimisation problems for eigenvalues. For the first time a systematic study of such properties of spectral problems is performed in view of direct applications to shape optimisation and inverse problems.

§2. Constructing the asymptotics.

2.1. **First term of boundary layer type.** Let $\lambda^0 = \lambda^0_m$ be a simple eigenvalue for problem (1.8) and $v^0 = v^0_m$ the corresponding eigenfunction, normalised by condition (1.9). The Laplace operator Δ_x in the curvilinear coordinates (n, s) takes the form

$$J(n,s)^{-1}\partial_n J(n,s)\partial_n + J(n,s)^{-1}\partial_s J(n,s)^{-1}\partial_s \tag{2.12}$$

where $J(n, s) = 1 + n\varkappa(s)$ is the Jacobian, and $\varkappa$ stands for the curvature of the curve Γ. Under the transformation to the rapid variables $\xi = (\xi_1, \xi_2)$, the splitting occurs

$$\Delta_x = h^{-2}\Delta_\xi + h^{-1}\varkappa(O)(\partial_{\xi_1} - 2\xi_1\partial^2_{\xi_2}) + \dots \tag{2.13}$$

In the coordinates (n, s) the gradient ∇_x takes the form $(\partial_n, J(n, s)^{-1}\partial_s)$, and the projection n^h_n, n^h_s of the unit normal vector n^h onto the coordinate axes for the variables n and s are given by the formulae

$$n^h_n = d^{-1/2}J^{-1}\nu_1, \quad n^h_s = d^{-1/2}J^{-1}\nu_2, \quad d = \nu_1^2 + J^{-2}\nu_2^2, \tag{2.14}$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit vector on the boundary $\partial\Xi \subset \mathbb{R}^2$.

Therefore, denoting by ∂_ν the directional derivative along the normal vector ν, we obtain in the rapid coordinates the decomposition

$$\partial_{n^h} = d^{-1/2}(\nu_1\partial_n + J^{-2}\nu_2\partial_s) = h^{-1}\partial_\nu + \varkappa(O)\xi_1(\nu_2\partial_\nu - 2\partial_{\xi_2}) + \dots \tag{2.15}$$

In (2.13) and (2.15) by dots are denoted the terms which are not important for the subsequent analysis. Taking into account the homogeneous Neumann condition in (1.8), the function v^0 in Ch-neighbourhood of the point O admits the representation

$$\begin{aligned} v^0(x) &= v^0(O) + s\partial_s v^0(O) + \frac{1}{2}\left(n^2\partial^2_n v^0(O) + s^2\partial^2_s v^0(O)\right) + O((n^2+s^2)^{3/2}) \\ &= v^0(O) + h\xi_2\partial_s v^0(O) + \frac{1}{2}h^2\left(\xi_1^2\partial^2_n v^0(O) + \xi_2^2\partial^2_s v^0(O)\right) + O(h^3). \end{aligned} \tag{2.16}$$

Replacing the eigenvalue and the eigenfunction in (1.3), (1.4) by ansätze (1.10) and (1.11), taking into account relations (2.13), (2.15) and collecting the terms of order h^{-1} in the equation, and of order h^0 in the boundary conditions, which are written in rapid variables, we arrive at the problem

$$-\Delta_\xi w^1(\xi) = 0, \quad \xi \in \Xi, \quad \partial_\nu w^1(\xi) = -\nu_2(\xi)\partial_s v^0(O), \quad \xi \in \partial\Xi\,. \tag{2.17}$$

We recall the well known formulae

$$\int\limits_{\partial\omega\cup\partial\Xi} \nu_2(\xi)ds_\xi = 0, \qquad \int\limits_{\partial\omega\cup\partial\Xi} \xi_j\nu_k ds_\xi = -\delta_{j,k} mes_2\omega, \quad j,k = 1,2. \tag{2.18}$$

The first formula shows that the right-hand side of the boundary condition in (2.17) has the null integral over the curve $\partial\omega \cup \partial\Xi$, we note that $\nu_2 = 0$ on the straight-line part $\partial\Xi \setminus \partial\omega$ of the boundary. In this way, there exists a unique generalized solution $w^1 \in H^1_{loc}(\overline{\Xi})$ of problem (2.17), decaying at infinity. The solution is represented in the form

$$w^1(\xi) = \partial_s v^0(O)W(\xi), \tag{2.19}$$

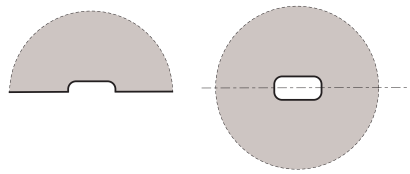

FIGURE 3. The domain Ξ and its extension Ξ^{00}

where W is a canonical solution of the Neumann problem

$$-\Delta_\xi W(\xi) = 0, \quad \xi \in \Xi, \quad \partial_\nu W(\xi) = -\nu_2(\xi), \quad \xi \in \partial\Xi, \tag{2.20}$$

which admits the representation

$$W(\xi) = \frac{\mathbf{m}}{\pi}\frac{\xi_2}{\rho^2} + O(|\xi|^{-2}) = \frac{\mathbf{m}}{\pi}\rho^{-1}\sin\varphi + O(\rho^{-2}), \tag{2.21}$$

in the system of polar coordinates (ρ, φ), with $\rho = |\xi|$ and $\varphi \in (-\pi/2, \pi/2)$.

We evaluate the coefficient $\mathbf{m}$ by applying method [29]. To this end, the Green formula on the set $\Xi_R = \{\xi \in \Xi : \rho < R\}$ is used with the functions W and $Y = \xi_2 + W$ since Y turns out to be a solution, with the growth at infinity, of the homogeneous problem (2.20). We have

$$\int\limits_{\partial\omega\cup\partial\Xi} Y\partial_\nu W ds_\xi = \int\limits_{\{\xi\in\mathbb{R}^2_-:\rho=R\}} (W\partial_\rho Y - Y\partial_\rho W) ds_\xi =$$
$$= \frac{\mathbf{m}}{\pi}\int\limits_{-\pi/2}^{\pi/2}\Big(\rho^{-1}\sin\varphi(\sin\varphi) - \rho\sin\varphi(-\rho^{-2}\sin\varphi)\Big)R|_{\rho=R}d\varphi + O(R^{-1}) = \mathbf{m} + O(R^{-1}). \tag{2.22}$$

On the other hand, in view of (2.18), the following relation is valid

$$\int\limits_{\partial\omega\cup\partial\Xi} Y\partial_\nu W ds_\xi = \int\limits_{\partial\omega\cup\partial\Xi} W\partial_\nu W ds_\xi - \int\limits_{\partial\omega\cup\partial\Xi}\xi_2\nu_2 ds_\xi = \int\limits_{\Xi}|\nabla_\xi W|^2 d\xi + mes_2\omega, \tag{2.23}$$

where $mes_2\omega$ denotes the area of ω. Therefore, the limit passage $R \to \infty$ leads to the relation

$$\mathbf{m}(\Xi) := \mathbf{m} = \int\limits_{\Xi}|\nabla_\xi W(\xi)|^2 d\xi + mes_2\omega. \tag{2.24}$$

Remark 2.1. As it is indicated in section 1.2, the approach with even extension of a harmonic function over the boundary with the homogeneous Neumann condition, is applicable to the function W (see Fig. 3). As a result, problem (2.20) can be transformed to the exterior Neumann problem in the domain

$$\Xi^{00} = \{\xi = (\xi_1, \xi_2) \in \mathbb{R} : (-|\xi_1|, \xi_2) \notin \overline{\omega}\}. \tag{2.25}$$

In this way, the extended function W becomes a solution, to exactly the same problem as introduced in monograph [41] for the description of the virtual mass tensor. Hence, $\mathbf{m}(\Xi)$ is twice the bottom right-hand element of the associated virtual mass matrix. ■

2.2. **Second boundary layer term.** The right-hand side of the problem

$$-\Delta_\xi w^2(\xi) = F^2(\xi), \quad \xi \in \Xi, \quad \partial_{nu} w^2(\xi) = G^2(\xi), \quad \xi \in \partial\Xi, \tag{2.26}$$

for the next term of the boundary layer type in ansatz (1.11) can be determined using formulae (2.13), (2.15) and (2.1). Indeed, assuming that solutions to (1.3), (1.4) take the form of ansätze (1.10) and (1.11) and offer splitting out the terms of order h^0 in the equation as well as of order h^1 in the boundary conditions, written in the rapid variables, we find that

$$F^2(\xi) = \varkappa(O)\left(\partial_{\xi_1} w^1(\xi) - 2\xi_1 \partial^2_{\xi_2} w^1(\xi)\right), \tag{2.27}$$

and

$$\begin{aligned} G^2(\xi) \quad &= \quad -\varkappa(O)\xi_1\nu_2(\xi)\left(\partial_\nu w^1(\xi) + \nu_2(\xi)\partial_s v^0(O)\right) \\ &+ \quad 2\varkappa(O)\xi_1\nu_2(\xi)\left(\partial_{\xi_2} w^1(\xi) + \partial_s v^0(O)\right) \\ &- \quad \left(\xi_1\nu_1(\xi)\partial^2_n v^0(O) + \xi_2\nu_2(\xi)\partial^2_s v^0(O)\right) \\ &=: \quad G^2_1(\xi) + G^2_2(\xi) + G^2_3(\xi). \end{aligned} \tag{2.28}$$

We note immediately that $G^2_1 = 0$ in view of the boundary conditions in problem (2.17). By formulae (2.19) and (2.21) the following expansion holds true

$$F^2(\xi) = \pi^{-1}\mathbf{m}\varkappa(O)\partial_s v^0(O)\rho^2\left[2\sin(4\varphi) + \sin(2\varphi)\right] + O(\rho^{-3}), \quad \rho \to \infty. \tag{2.29}$$

The function

$$\xi \mapsto U^2(\xi) = -(8\pi)^{-1}\mathbf{m}\varkappa(O)\partial_s v^0(O)\left[\sin(4\varphi) + 2\sin(2\varphi)\right] \tag{2.30}$$

turns out to be harmonic in the half-plane $\mathbb{R}^2_-$ and verifies the homogeneous Neumann conditions everywhere on the line $\partial\mathbb{R}^2_-$ except of the point $\xi = 0$. The function compensates the leading term of asymptotics of the right-hand side (2.29) of the Poisson equation in problem (2.26), function (2.30) participates in the expansion of its solution at infinity

$$w^2(\xi) = \frac{a}{\pi}\ln\rho + c - (8\pi)^{-1}\mathbf{m}\varkappa(O)\partial_s v^0(O)\left[\sin(4\varphi) + 2\sin(2\varphi)\right] + O(\rho^{-1}). \tag{2.31}$$

Here c stands for a generic constant, we assume that $c = 0$, and the logarithmic term is included since, in advance, it is not clear of there is a solution to problem (2.26) in the class of bounded functions; on the other hand it is known, see e.g., [36, Ch. 2] that there exist a solution with the logarithmic growth for $\rho \to 0$ and that such a solution is determined up to an additive constant.

Remark 2.2. Denotation $z(\xi) = z_0(\xi) + O(\rho^{-p})$ used in (2.21) and (2.29), (2.31) means that

$$z(\xi) = z_0(\xi) + \widetilde{z}(\xi), \quad |\nabla^q_\xi \widetilde{z}(\xi)| \le c_q\rho^{-p-q}, \quad q = 0, 1, \ldots, \quad \rho = |\xi| \le R_0; \tag{2.32}$$

where $\nabla^q_\xi \widetilde{z}$ is the collection of all order q derivatives of the function $\widetilde{z}$, and the radius R_0 is selected in such a way that $\overline{\omega} \subset \{\xi : \rho < R_0\}$. For a solution w^1 of problem (2.20) the estimates of form (2.32) for the remainder $\widetilde{w}^1$ are straightforward, since the remainder verifies the Laplace equation with the Neumann boundary conditions on the sets $\{\xi \in \mathbb{R}^2 : \rho > R_0\}$ and $\{\xi \in \partial\mathbb{R}^2_- : \rho > R_0\}$, respectively. For such an equation, e.g., the Fourier method can be used in order to obtain the representation of the solution in the form of convergent series, with decaying at infinity harmonic functions. In problem (2.26) there is non-trivial right-hand side of the equation, therefore by general theory of elliptic boundary value problems in domains with conical points and outlets to infinity, we refer to the key papers [17], [29], [30] and e.g., to monograph [36], in the decomposition of the solution (in the form of a series) the logarithmic multipliers can occur beside the Poisson kernel $\pi^{-1}\ln\rho$. The direct evaluation of the function U^2 shows, that the principal term of asymptotics (2.29) of the right-hand side F^2 for the equation in problem (2.26) does not lead to appearance of $\ln\rho$. In a similar manner it can be verified that the following asymptotic terms in the expressions

$$W(\xi) = c_1\rho^{-1}\sin\varphi + c_2\rho^{-2}\cos(-2\varphi) + O(\rho^{-3}),$$

$$F^2(\xi) = C_1\rho^{-2}\sin(4\varphi) + C_2\rho^{-3}(3\cos(5\varphi) + \cos(3\varphi)) + O(\rho^{-4})$$

do not lead to the appearance of logarithms: the term with the coefficient C_2 is compensated by the function $\xi \longmapsto -\frac{1}{8}C_2\rho^{-1}(\cos(5\varphi) + \cos(3\varphi) + \cos\varphi)$ which enjoys the properties of function (2.30). Estimate of the remainder in representation (2.31), (2.32) for the solution w^2 are justified again by the general theory (see [30]). ■

For the computation of the multiplier a in expansion (2.31) the method of [30] is used again. Actually, we inject the functions w^2 and w^1 in the Green formula on the set Ξ_R and compute the integral on the semi-circle of the radius R taking into account expansion (2.31):

$$\begin{aligned}\int\limits_{\Xi_R} F^2(\xi)d\xi + \int\limits_{\partial\omega\cap\partial\Xi} G^2(\xi)ds_\xi = -\int\limits_{\{\xi\in\mathbb{R}^2_-:\rho=R\}} \partial_\rho w^2(\xi)ds_\xi = \\ -\int\limits_{-\pi/2}^{\pi/2}\left(\frac{a}{\pi} + \frac{1}{4\pi}\mathbf{m}\varkappa(O)\partial_s v^0(O)(\sin 2\varphi - 2\varphi)\right)d\varphi + O(R^{-1}) = -a + O(R^{-1}).\end{aligned} \tag{2.33}$$

It remains to study the integral from the left-hand of (2.33).

By equality (2.18), the form of Laplace operator in curvilinear coordinates given in (2.12) and relation (1.8) for the function v^0, we find

$$\int\limits_{\partial\omega\cap\partial\Xi} G_3^2(\xi)ds_\xi = mes_2\omega(\partial_n^2 v^0(O) + \partial_s^2 v^0(O)) = -\lambda^0 v^0(O)mes_2\omega. \tag{2.34}$$

Beside that, in view of equalities (2.18) it follows that

$$\int\limits_{\partial\omega\cap\partial\Xi} \varkappa(O)\xi_1\nu_2(\xi)\partial_s v^0(O)ds_\xi = 0.$$

The Stokes formula yields

$$\int\limits_{\Xi_R}\frac{\partial w^1}{\partial\xi_1}(\xi)ds_\xi = \int\limits_{\partial\omega\cap\partial\Xi}\nu_1(\xi)w^1(\xi)ds_\xi + \int\limits_{\{\xi\in\mathbb{R}^2_-:\rho=R\}}\rho^{-1}\xi_1 w^1(\xi)ds_\xi\ .$$

Since the leading asymptotic term of the solution w^1 is an odd function in the variable ξ_2, the integral over the half-circle of the radius R is of the order $O(R^{-1})$ (the integral over $(-\pi/2,\pi/2)\ni\varphi$ of an odd function in $\xi_2 = \rho\sin\varphi$ vanishes). In view of the Green formula, the integral over the curve $\partial\omega\cap\partial\Xi$ equals

$$\int\limits_{\partial\omega\cap\partial\Xi} w^1(\xi)\frac{\partial\xi_1}{\partial\nu}ds_\xi = \int\limits_{\partial\omega\cap\partial\Xi}\xi_1\frac{\partial w^1}{\partial\nu}(\xi)ds_\xi + \int\limits_{\{\xi\in\mathbb{R}^2_-:\rho=R\}}\left(\xi_1\partial_\rho w^1(\xi) - w^1(\xi)\partial_\rho\xi_1\right)ds_\xi.$$

In the limit for $R\to\infty$, the integral over the half-circle vanishes by the same argument as above, and the integral over the curve $\partial\omega\cap\partial\Xi$ becomes zero due to the boundary conditions in problem (2.17) and to the second equality (2.18) for $j = 1$, $k = 2$. Finally,

(2.35)
$$-2\varkappa(O)\int\limits_{\Xi_R}\xi_1\frac{\partial^2 w^1}{\partial\xi_2^2}(\xi)d\xi$$
$$= -2\varkappa(O)\int\limits_{\partial\omega\cap\partial\Xi}\xi_1\nu_2(\xi)\frac{\partial w^1}{\partial\xi_2}(\xi)ds_\xi - 2\varkappa(O)\int\limits_{\{\xi\in\mathbb{R}^2_-:\rho=R\}}\rho^{-1}\xi_1\xi_2\frac{\partial w^1}{\partial\xi_2}(\xi)ds_\xi\ .$$

The latter integral equals to $O(R^{-1})$, hence the leading term of the order $O(\rho^{-1})$ of asymptotics for the expression $\xi_2\partial w^1/\partial\xi_2$ with $\rho\to\infty$ is still odd with respect to the variable ξ_2, therefore it is annulated by integration. The first integral, without the minus sing, from the right-hand side in (2.35) is present in the integral of the function G_2^2 over $\partial\omega\cap\partial\Xi$, see definition in (2.28). Recalling that $G_1^2 = 0$ and collecting the obtained formulae, we can conclude in accordance with (2.34) that the limit passage $R\to\infty$ in relation (2.33) results in the equality

(2.36)
$$a = \lambda^0 v^0(O) mes_2\omega.$$

We point out, that the curvature $\varkappa(O)$ appears in several integrals, which finally cancel each other.

2.3. **The correction term of regular type.** For the terms of boundary layer type, the asymptotics can be written in the condensed form

(2.37)
$$w^q(\xi) = t^q(\xi) + O(\rho^{q-3}),\quad \rho\to\infty,\quad q = 1,2.$$

Outside a small neighbourhood of the point O we have:

(2.38)
$$hw^1(\xi) + h^2w^2(\xi) = h^2\left(t^1(n,s) + t^2(n,s) - a\pi^{-1}\ln h\right) + O(h^3) =: h^2T(x,\ln h) + O(h^3).$$

In view of the multiplier h^2 the expression for T should be present in the problem for the term v^2 of regular type

(2.39)
$$-\Delta_x v^2(x) = \lambda^0 v^2(x) + \lambda' v^0(x) + f^2(x),\quad x\in\Omega,$$

(2.40)
$$\partial_n v^2(x) = g^2(x),\quad x\in\Gamma.$$

The first two terms in the right-hand side of (2.39) are obtained as a result of replacement in (1.3) of eigenvalues and eigenfunctions by ansätze (1.10), (1.11) and collection of order h^2 terms written in the slow variables x. The right-hand side g^2 of the boundary condition (2.40) is but the discrepancy which results from the multiplication of the boundary layer terms with the cut-off function χ. If it is assumed that in the vicinity of the boundary the cut-off function χ depends only on the tangential variable s, and it is independent of the normal variable n, than it follows that $g^2 = 0$, since in problems (2.17) and (2.26) the boundary conditions on $\partial\Xi\setminus\partial\omega$ are homogeneous. It is clear that such a requirement can be readily satisfied. Thus, we further assume $g^2 = 0$.

Multiplication by the cut-off function introduces in equation (2.39) the discrepancies of the boundary layer terms. However, the commutators of the operator Δ_x with the cut-off function χ, are not the only source of terms in the function f^2. Actually, the application of the procedure described in sections 2.1 and 2.2 leads to the right-hand side of the Poisson equation for the subsequent term w^3 of ansatz (1.11), the term is of order $O(\rho^{-1})$, i.e., the solution of such a problem enjoys at least the linear growth at infinity. As a result, the main term of asymptotics for the expression $h^3w^3(h^{-1}n, h^{-1}s)$ contributes in relation (2.38) with a term of order h^2, which means that without construction of w^3 it is impossible to determine

v^2, and the algorithm becomes incorrect. To avoid such a contradiction, in paper [25] (see also [28, introductory Ch. 2]) is proposed the procedure of rearrangement of discrepancies which make it possible to define, in the framework of compound asymptotic expansions, a problem for v^2 using only the terms v^0 and w^1, w^2 of ansatz (1.11). This approach consists in the analysis of terms depending on w^1 and w^2 from equation (1.3). In particular, the terms which do not enjoy the sufficient decay rate $O(\rho^{-2})$ at infinity, are rewritten in slow variables and are included in the problem for the terms of regular type (simple examples of such a procedure can be found in [28, Ch. 2] and papers [32], [4], the general framework is described in [28, Ch. 4]). In our case the detached parts are given by the expressions

$$f^2(x) = \lambda^0\chi(x)T(x, \ln h) + \Delta_x(\chi(x)T(x, \ln h)). \tag{2.41}$$

We verify that the function f^2, smooth outside of a neighbourhood of the origin O, is of the growth $O(|x|^{-1})$ for $x \to O$ which means that f^2 belongs to $H^{-1}(\Omega)$ and is admissible for the right-hand side of equation (2.39). For the first term this fact is obvious, since $t^1(n, s) = O(|x|^{-1})$ and $t^0(n, s) = O(|\ln|x||)$. Let us consider the second term. Representation (2.13) of the operator Δ_x in the curvilinear coordinates:

$$\begin{aligned}\Delta_x = (\partial_n^2 + \partial_s^2) + \varkappa(O)(\partial_n - n\partial_s^2) + a_{12}(n, s)\partial_n^2 + a_{22}(n, s)\partial_s^2 + \\ a_{11}(n, s)\partial_n + a_{21}(n, s)\partial_s =: L^0(\partial_n, \partial_s) + L^1(n, \partial_n, \partial_s) + L^2(n, s, \partial_n, \partial_s).\end{aligned} \tag{2.42}$$

Here a_{jk} are smooth functions in a neighbourhood of the point O, in variables n and s, in addition

$$a_{jk}(0, 0) = 0, \quad \partial_n a_{j2}(0, 0) = 0, \quad \partial_s a_{j2}(0, 0) = 0, \quad j, k = 1, 2.$$

Therefore,

$$\Delta_x T = J^{-1}\left\{L^0t^1 + (L^0t^2 + L^1t^0) + L^1t^2 + L^2(t^1 + t^2)\right\}. \tag{2.43}$$

The first two terms in braces vanishes by the definition of the singular components t^1 and t^2 and the terms L^1t^2 and $L^2(t^1+t^2)$ are of the required order. Thus, $g^2 = 0$ and $|x|^\mu f^2 \in L_2(\Omega)$ for any $\mu > 0$.

Under the assumption that λ^0 is a simple eigenvalue, problem (2.39), (2.40) with such right-hand sides admits a solution v^2 in the Sobolev space $H^1(\Omega)$ if and only if the following relation is satisfied

$$\lambda'(v^0, v^0)_\Omega + (f^2, v^0)_\Omega = 0. \tag{2.44}$$

Owing to the normalisation condition (1.9), the coefficient of λ' equals one. Integral of the product f^2v^0 is convergent, which means that

$$(f^2, v^0)_\Omega = \lim_{\delta\to+0}\int_{\Omega_\delta}(\lambda^0T + \Delta_xT)v^0dx, \tag{2.45}$$

where $\Omega_\delta = \Omega\setminus\left\{x : n^2 + s^2 \le \delta^2\right\}$. The arc $\gamma_\delta = \partial\Omega\setminus\partial\Omega_\delta$ turns out to be a half-circle in the curvilinear coordinate system. We imitate the polar coordinate system in the curvilinear coordinates by putting $n = r\cos\varphi$ and $s = r\sin\varphi$ while calling in the sequel ρ, φ, with $\rho = h^{-1}r$, the polar coordinate system. Integration by parts with the Green formula in Ω_δ for the smooth functions T and v^0 in the domain yields

$$\int_{\Omega_\delta} f^2v^0dx = \int_{\gamma_\delta}(v^0\partial_NT - T\partial_Nv^0)ds_x. \tag{2.46}$$

Let us observe that $ds_x = d(n,s)^{1/2}J(n,s)ds_x$ on the curve γ_δ, and owing to formulae (2.14) the derivative ∂_N along the normal to the contour γ_δ satisfies the relation

$$\partial_N T = N_n \partial_n T + N_s J^{-1}\partial_s T,$$

$$N_n = d^{-1/2}cos\varphi, \quad N_s = d^{-1/2}J^{-1}sin\varphi, \quad d = (cos\varphi)^2 + J^{-2}(\sin\varphi)^2.$$

We take into account only the terms with the non-null limits for $\delta \to +0$; it follows that expression (2.46) takes the form

$$\partial_s v^0(O)\delta \int_{-\pi/2}^{\pi/2} \left(s\partial_r t^1(n,s) - t^1(n,s)\partial_r s\right)\Big|_{r=\delta} d\varphi$$

$$+v^0(O)\varkappa(O) \int_{-\pi/2}^{\pi/2} n\left(n\partial_n t^1(n,s) - s\partial_s t^1(n,s)\right)\Big|_{r=\delta} d\varphi$$

$$+v^0(O)\delta \int_{-\pi/2}^{\pi/2} \partial_r t(n,s)\Big|_{r=\delta} d\varphi + o(1).$$

The second term, which depends on the curvature $\varkappa(O)$, vanishes (note that the integral is zero since $n\partial_n t^1 - s\partial_s t^1$ is still odd function of the variable s). The first integral is evaluated in (2.22), and the third integral is computed in (2.33). In view of relations (2.19) and (2.36) the integrals equal to $-\partial_s v^0(O)m\partial_s v^0(O)$ and $v^0(O)a = \lambda^0|v^0(O)|^2 mes_2\omega$, respectively. Thus, by the limit passage in (2.45), we obtain from compatibility condition (2.44) for problem (2.39), (2.40), the formula for asymptotic correction term in representation (1.10) of the simple eigenvalue

$$\lambda'_m = \mathbf{m}(\Xi)|\partial_s v^0_m(O)|^2 - \lambda^0 mes_2\omega|v^0_m(O)|^2. \tag{2.47}$$

2.4. **Multiple eigenvalues.** Assume now, that λ^0_m is an eigenvalue of the multiplicity $\varkappa_m > 1$, i.e.,

$$\lambda^0_{m-1} < \lambda^0_m = \cdots = \lambda^0_{m+\varkappa_m-1} < \lambda^0_{m+\varkappa_m}. \tag{2.48}$$

In such a case ansätze (1.10) and (1.11) are valid for $p = m,\ldots,m+\varkappa_m-1$, however, the principal terms of expansions for the eigenfunctions $u^h_m,\ldots,u^h_{m+\varkappa_m-1}$ in problem (1.3), (1.4) are predicted in the form of linear combinations

$$v^{p0} = a^p_1 v^0_m + \cdots + a^p_{\varkappa_m} v^0_{m+\varkappa_m-1} \tag{2.49}$$

of eigenfunctions corresponding in problem (1.8) to the eigenvalue λ^0_m, subject to the orthogonality and normalisation conditions (1.9). The coefficients of columns $a^p = (a^p_1,\ldots,a^p_{\varkappa_m})$ in (2.49) are to be determined. Under assumption that the columns $a^m,\ldots,a^{m+\varkappa_m-1}$ are unit vectors and are pairwise orthogonal, i.e.,

$$a^p \cdot a^q = \delta_{p,q}, \quad p,q = m,\ldots,m+\varkappa_m-1, \tag{2.50}$$

then the linear combinations (2.49) with $p = m,\ldots,m+\varkappa_m-1$, are simply a new orthonormal basis in the eigenspace of the eigenvalue λ_m.

The construction of boundary layer terms is performed in the same way as in the previous section. When solving problem (2.39), (2.40) for the regular term v^{p2}, it appears $\varkappa_m$ compatibility conditions

$$\lambda^{p\prime}(v^{p0}, v^0_{m+k})_\Omega + (f^{p2}, v^0_{m+k})_\Omega = 0, \quad k = 0,\ldots,\varkappa_m-1, \tag{2.51}$$

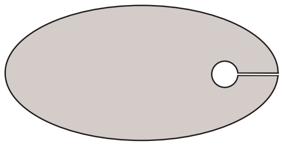

FIGURE 4. The geometry which makes the extension in H^1 impossible.

which can be written in the form of a linear system of $\varkappa_m$ algebraic equations

$$\lambda^{p\prime} a^p = \mathbf{M} a^p \tag{2.52}$$

with the matrix $\mathbf{M} = (\mathbf{M}_{ik})_{j,k=0}^{\varkappa_m - 1}$ of the size $\varkappa_m \times \varkappa_m$,

$$\mathbf{M}_{jk} = \mathbf{m}(\Xi)\partial_s v^0_{m+k}(\mathcal{O})\partial_s v^0_{m+j}(\mathcal{O}) - \lambda^0_m v^0_{m+k}(\mathcal{O}) v^0_{m+j}(\mathcal{O}) mes_2(\omega). \tag{2.53}$$

Formula (2.53) is derived in exactly the same way as it is for formula (2.47) The matrix $\mathbf{M}$ is symmetric, and its real eigenvalues $\lambda^{m\prime}, \dots, \lambda^{m+\varkappa_m-1\prime}$ corresponds to eigenvectors $a^m, \dots, a^{m+\varkappa_m-1}$, which satisfy conditions (2.50). Actually, just these attributes of the matrix $\mathbf{M}$ with elements (2.53) are included in ansätze (1.10) and (1.11), (2.49) for eigenvalues λ^h_p and eigenfunctions u^h_p of problem (1.3), (1.4) for $p = m, \dots, m + \varkappa_m - 1$ in case (2.48).

§3. JUSTIFICATION OF ASYMPTOTICS

3.1. **The weighted Poincaré inequality.** The subspace $H^1(\Omega(h))_\perp$ of the Sobolev space $H^1(\Omega(h))$ contains functions of zero mean over the set $\Omega(h)$.

Lemma 3.1. *The following inequality is valid*

$$\|u; L_2(\Omega(h))\| \le c\|r^{-1}(1 + |\ln r|)^{-1} u; L_2(\Omega(h))\| \le C\|\nabla_x u; L_2(\Omega(h))\|, \tag{3.1}$$

where $r(x) = dist(x, \mathcal{O})$, and the constant c is independent of the parameter $h \in (0, h_0]$ and the function $u \in H^1(\Omega(h))_\perp$.

Proof. In the representation

$$u(x) = u_*(x) + b_* \tag{3.2}$$

the constant b_* is chosen such that

$$\int_{\Omega_*} u_*(x)dx = 0, \quad b_* = -(mes_2\Omega_*)^{-1} \int_{\Omega_*} u(x)dx \tag{3.3}$$

where the domain $\Omega_* \subset \Omega$ satisfies $\Omega_* \neq \varnothing$ and $\Omega_* \cap \omega_h = \varnothing$ for $h \in (0, h_0]$. Let us construct an extension $\widehat{u}_*$ of the function u_* in the class H^1, from the set $\Omega \setminus \mathbb{B}_{Rh}$ onto Ω, in such a way that the estimate is valid

$$\|\nabla_x \widehat{u}_*; L_2(\Omega)\| \le c\|\nabla_x u_*; L_2(\Omega_{Rh})\| = c\|\nabla_x u; L_2(\Omega_{Rh})\| \le c\|\nabla_x u_*; L_2(\Omega(h))\|. \tag{3.4}$$

Here $\mathbb{B}_{Rh}$ is the ball of radius Rh and the centre $\mathcal{O}$, and $\omega_h \subset \mathbb{B}_{Rh}$.

We emphasise that an extension from $\Omega(h)$ onto Ω may not exist in the class H^1, for example in the case of a crack, see Fig. 4 and section 4.3. Stretching coordinates $x \longmapsto \eta = h^{-1}x$ transforms the set $\Sigma_{Rh} = \{x \in \Omega : Rh > r > Rh/2\}$ into the half-ring $\Sigma(h)$ with fixed radia and gently sloped ends, we recall that the boundary $\partial\Omega$ is smooth. Therefore, for the component $U_\perp$ of the similar decomposition to (3.2), (3.3)

$$U_*(\eta) = U_\perp(\eta) + b_\perp, \quad \int\limits_{\Sigma(h)} U_\perp(\eta)d\eta = 0 \tag{3.5}$$

of the function $\eta \longmapsto U_\perp(\eta) = u_\perp(x)$, the Poincaré inequality is valid

$$\|U_\perp; L_2(\Sigma(h))\| \le c\|\nabla_\eta U_\perp; L_2(\Sigma(h))\| = c\|\nabla_\eta U_*; L_2(\Sigma(h))\|$$

and there exists an extension denoted by $\widehat{U}_\perp$ from $\Sigma(h)$ onto $\Sigma^0(h) = \{\eta : x \in \Omega, \tau < Rh\}$, such that

$$\|\widehat{U}_\perp; H^1(\Sigma^0(h))\| \le c\|U_\perp; H^1(\Sigma(h))\| \le c\|\nabla_\eta U_\perp; L_2(\Sigma(h))\|.$$

In these inequalities, the factors c are independent of $U_\perp$ and $h \in (0, h_0]$.

The required extension $\widehat{u}_*$ is defined as follows

$$\widehat{u}_*(x) = \begin{cases} u_*(x), & x \in \Omega \setminus \mathbb{B}_{Rh}, \\ \widehat{U}_\perp(\eta) + c_\perp, & x \in \Omega \cap \mathbb{B}_{Rh}. \end{cases} \tag{3.6}$$

Owing to the above relations, we have

$$\|\nabla_x \widehat{u}_*; L_2(\Omega)\| \le c\|\nabla_x u_* : L_2(\Omega \setminus \mathbb{B}_{Rh})\| \le c\|\nabla_x u; L_2(\Omega(h))\|.$$

For function (3.6), in the same way as before, the orthogonality condition from (3.3) is satisfied, which means that

$$\|\widehat{u}_*; L_2(\Omega)\| \le c\|\nabla_x \widehat{u}_* : L_2(\Omega)\| \le c\|\nabla_x u; L_2(\Omega(h))\|. \tag{3.7}$$

By the one-dimensional Hardy inequality

$$\int\limits_0^1 r^{-1}|\ln r|^{-2} Z(r)^2 dr \le 4 \int\limits_0^1 r \left|\frac{dZ}{dr}(r)\right|^2 dr, \quad Z \in C^1_c[0,1),$$

the following estimate is valid

$$\|r^{-1}(1 + |\ln r|)^{-1}\widehat{u}_* : L_2(\Omega)\| \le (\|\nabla_x \widehat{u}_*; L_2(\Omega_*)\|) \le c\|\nabla_x u; L_2(\Omega(h))\|. \tag{3.8}$$

For the constant $b_\perp$ in decomposition (3.5) we now obtain

$$\begin{aligned} |b_\perp| &= \Big| \int\limits_{\Sigma(h)} U_*(\eta)d\eta \Big| \le c\|U_*; L_2(\Sigma(h))\| = c\|\widehat{U}_{ast}; L_2(\Sigma(h))\| \\ &= ch^{-1}\|\widehat{u}_*; L_2(\Sigma_{Rh})\| \le c(1 + |\ln h|)\|r^{-1}(1 + |\ln r|)^{-1}\widehat{u}_*; L_2(\Sigma_{Rh})\|. \end{aligned}$$

Beside that, the image $\Sigma_\omega(h)$ of the set $\Omega(h) \cap \mathbb{B}_{Rh}$ under stretching of coordinates, again enjoys the gently sloped boundary, hence

$$\|U_*; L_2(\Sigma_\omega(h))\| \le c(\|\nabla_\eta U_*; L_2(\Sigma_\omega(h))\| + \|U_*; L_2(\Sigma(h))\|).$$

In this way we have

$$\begin{aligned} &\|r^{-1}(1 + |\ln r|)^{-1}u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| \\ &\le ch^{-1}(1 + |\ln h|)^{-1}\|u_* : L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| \\ &= ch^{-1}(1 + |\ln h|)^{-1}\|U_\perp + b_\perp : L_2(\Sigma_\omega(h))\| \\ &\le c(1 + |\ln h|)^{-1}(\|\nabla_\eta U_*; L_2(\Sigma_\omega(h))\| + \|U_*; L_2(\Sigma(h))\| + |b_\perp|) \\ &\le c(\|\nabla_x u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| + \|r^{-1}(1 + |\ln r|)^{-1}\widehat{u}_*; L_2(\Sigma_{Rh})\|). \end{aligned}$$

The relation combined with relation (3.8) and definition (3.6) shows that for the component u_* of decomposition (3.2), the required inequality (3.1) is verified. Since $u \in H^1(\Omega(h))_\perp$, the constant b_* in (3.2) satisfies the estimate

$$|b_*| = \Big| \int\limits_{\Omega(h)} (u(x) - u_*(x))dx \Big| = \Big| \int\limits_{\Omega(h)} u_*(x)dx \Big| \tag{3.9}$$
$$\leq c\|u_*; L_2(\Omega(h))\| \leq c\|\nabla_x u; L_2(\Omega(h))\|.$$

It remains to note that the integral

$$\int\limits_{\Omega(h)} r^{-2}(1 + |\ln r|)^{-2}dx$$

with the weight factor, present in the left-hand side of (3.1) does not exceed a constant independent of the parameter h. ■

In the sequel, the left-hand side of formula (3.1) is denoted by $|||u; \Omega(h)|||$.

In the proof of Lemma 3.1, it is constructed an extension denoted by $\widehat{u} := \widehat{u}_* + b_*$ of the function $u \in H^1(\Omega(h))_\perp$ from the set $\Omega(h) \setminus \mathbb{B}_{Rh}$ onto the domain Ω, such that

$$|||u; \Omega||| + \|\nabla_x \widehat{u}; L_2(\Omega)\| \leq c\|\nabla_x u; L_2(\Omega(h))\| \tag{3.10}$$

see formulae (3.7) and (3.9). Assume that $m \geq 1$ and $\widehat{u}^h_m$ is the extension described above of the eigenfunction u^h_m, for such an extension in view of relation (1.6) and the integral identity [21]

$$(\nabla_x u^h_m, \nabla_x z)_{\Omega(h)} = \lambda^h_m (u^h_m, z)_{\Omega(h)}, \quad z \in H^1(\Omega(h))_\perp, \tag{3.11}$$

which replaces problem (1.3), (1.4) for positive eigenvalues, the following relation is valid

$$\|\widehat{u}^h_m; H^1(\Omega)\|^2 \leq c\|\nabla_x u^h_m; L_2(\Omega(h))\|^2 = c\lambda^h_m. \tag{3.12}$$

The minimax principle (see e.g., [47]), where the test functions can be taken from the space $C^\infty_c(\Omega_*)$, show that for an arbitrary m there exist positive numbers h_m and c_m, such that

$$\lambda^h_m \leq c_m \quad \text{for} \quad h \in (0, h_m]. \tag{3.13}$$

Therefore, the norms $\|\widehat{u}^h_m; H^1(\Omega)\|$ are uniformly bounded with respect to the parameter $h \in (0, h_m]$ for a fixed m, i.e. the pairs $\left\{\lambda^h_m, \widehat{u}^h_m\right\}$ admit the weak limit $\left\{\widehat{\lambda}^0_m, \widehat{v}^0_m\right\} \in \mathbb{R} \times H^1(\Omega)$ for $h \to +0$ and the strong limit in $\mathbb{R} \times L_2(\Omega)$.

In the integral identity (3.11) we choose a test function $z \in C^\infty_c(\overline{\Omega} \setminus \mathcal{O})$ with the null mean value. For sufficiently small h, $\widehat{u}^h_m = u^h_m$ on the support of the function z, thus the limit passage in (3.11) leads to the equality

$$(\nabla_x \widehat{v}^0_m, z)_\Omega = \widehat{\lambda}^0_m (\widehat{v}^0_m, z)_\Omega. \tag{3.14}$$

Since $C^\infty_c(\overline{\Omega} \setminus \mathcal{O})$ is dense in $H^1(\Omega)$ (the elements of the Sobolev space $H^1(\Omega)$ have no traces at single points), by the completion argument, we can assume that in integral identity (3.14) the test function z belongs to $H^1(\Omega)_\perp$.

By inequalities (3.1) and (3.10), (3.11), it follows that

$$\Big| \int\limits_\Omega \widehat{u}^h_m dx - \int\limits_{\Omega(h)} u^h_m dx \Big| \leq \Big| \int\limits_{\Omega \cap \mathbb{B}_{Rh}} |\widehat{u}^h_m| dx + \int\limits_{\Omega(h) \cap \mathbb{B}_{Rh}} |u^h_m| dx \Big|$$
$$\leq ch^2(1 + |\ln h|)(|||\widehat{u}^h_m; \Omega||| + |||u^h_m; \Omega(h)|||)$$
$$\leq ch^2(1 + |\ln h|),$$

and

$$\left| \int_{\Omega} |\widehat{u}_m^h|^2 dx - \int_{\Omega(h)} |u_m^h|^2 dx \right| \leq ch(1 + |\ln h|).$$

Hence

$$\widehat{v}_m^0 \in H^1(\Omega), \quad \|\widehat{v}_m^0; L_2(\Omega)\| = 1,$$

i.e., $\widehat{\lambda}_m^0$ is an eigenvalue and $\widehat{v}_h^0$ is a normalised eigenfunction of problem (1.8).

Proposition 3.1. *Entries of sequences (1.5) and (1.7) are related by the limit passage*

$$\lambda_m^h \to \lambda_m^0 \quad \text{as} \quad h \to +0. \tag{3.15}$$

Proof is completed at the end of this section. We only observe that it has been already shown that $\lambda_m^h \to \lambda_p^0$, thus it suffices to prove that $p = m$.

From Lemma 3.1 it follows that the left-hand side of identity (3.11) can be chosen as the scalar product $\langle u_m^h, z\rangle$ in the space $H^1(\Omega(h))_\perp$. We define the operator K^h in the space $H^1(\Omega(h))_\perp$ by the formula

$$\langle K^h u, z\rangle = (u, z)_{\Omega(h)}, \quad u, z \in H^1(\Omega(h))_\perp. \tag{3.16}$$

It is easy to check that K^h is symmetric, positive and compact, therefore, self-adjoint. For $m \geq 1$ we set $\mu_m^h = (\lambda_m^h)^{-1}$. The positive eigenvalues and the corresponding eigenfunction of problem (1.3), (1.4) can be considered in an abstract framework, so we deal with the spectral equation in the Hilbert space $H = H^1(\Omega(h))_\perp$:

$$K^h u^h = \mu^h u^h. \tag{3.17}$$

The norm, defined by the scalar product $\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle$ is denoted by $\|\cdot\|_H$. The following statement [49] is known as lemma on *almost eigenvalues and eigenvectors*.

Lemma 3.2. *Let μ and $U \in H$ be such that $\|K^h U - \mu U\| = \tau$ and $\|U\|_H = 1$. Then there exists an eigenvalue μ_m^h of the operator K^h, which satisfies the inequality*

$$|\mu - \mu_m^h| \leq \tau.$$

Moreover, for any $\tau_\bullet > \tau$ the following inequality holds

$$\|U - U_\bullet\|_H \leq 2\tau/\tau_\bullet$$

where $U_\bullet$ is a linear combination of eigenfunctions of the operator K^h, associated to the eigenvalues from the segment $[\mu - \tau_\bullet, \mu + \tau_\bullet]$, $\|U_\bullet\|_H = 1$.

The asymptotic approximations μ and U of a solution to equation (3.16) is defined by the number $(\lambda_m^0 + h^2\lambda_m)^{-1}$ and by the function $\|V_m^h\|_H^{-1} V_m^h$, respectively, where $m \geq 1$ and λ_m' with V_m^h are, respectively, the correction given by (2.47) and the sum of the first four terms in the ansatz (1.11), found in §2. In the case of multiple eigenvalue λ_p^0, we take into consideration the specification provided at the end of section 2.4.

We estimate the quantity τ from Lemma 3.2. Since $\|V_m^h\| \geq \|v_m^0\|_H - c_m h$ and $\lambda_m^0 + h^2\lambda_m' \geq \lambda_m^0 - c_m h^2$, for a sufficiently small h it follows that

$$\begin{aligned} \tau &= \|K^h U - \mu U\|_H \\ &= (\lambda_m^0 + h^2\lambda_m')^{-1} \|V_m^h\|^{-1} \sup |\langle(\lambda_m^0 + h^2\lambda_m') K^h V_m^h, z\rangle| \\ &\leq c_m \sup |(\nabla_x V_m^h, \nabla_x z)_{\Omega(h)} - (\lambda_m^0 + h^2\lambda_m')(V_m^h, z)_{\Omega(h)}|, \end{aligned} \tag{3.18}$$

where the supremum is taken over the set $\{z \in H^1(\Omega(h))_\perp : \|z\|_H = 1\}$ and, hence, the L_2-norms of the test function z indicated in inequality (3.1), both standard and weighted, are bounded by a constant $\mathcal{N}$. Beside that, we can respect the proof of the trace theorem [21], which gives

$$\begin{aligned} h^{-1/2}(1+|\ln h|)^{1/2}\|z; L_2(\partial\omega_h\cap\Gamma(h))\| \\ \leq c|||z;\Omega(h)|||(\|\nabla_x z; L_2(\Omega(h))\| + |||z;\Omega(h)|||) \leq c\mathcal{N}. \end{aligned} \tag{3.19}$$

The last expression in (3.18) can be processed as follows

(3.20)
$$\begin{aligned} I &= I^1 + h^2 I^2 - h^4 I^3 + I^4 - I^5 - h^2 I^6 \\ &:= \Big((\nabla_x v_m^0, \nabla_x z)_{\Omega(h)} - \lambda_m^0 (v_m^0, z)_{\Omega(h)}\Big) + h^2\Big((\nabla_x v_m^2, \nabla_x z)_{\Omega(h)} - (\lambda_m^0 v_m^2 + \lambda_m' v_m^0, z)_{\Omega(h)}\Big) \\ &- h^4\lambda_m'(v_m^2, z)_{\Omega(h)} + \Big(\nabla_x\chi(hw_m^1 + h^2 w_m^2), \nabla_x z\Big)_{\Omega(h)} - \lambda_m^0\Big(\chi(hw_m^1 + h^2w_m^2), z\Big)_{\Omega(h)} \\ &- h^2\lambda_m'\Big(\chi(hw_m^1 + h^2 w_m^2), z\Big)_{\Omega(h)}. \end{aligned}$$

The estimates of two terms I^3 and I^6 are straightforward

$$\begin{aligned} |I^3| &\leq c_m\|v_m^2; L^2(\Omega)\|\mathcal{N}_z \leq c_m(1+|\ln h|)\mathcal{N}, \\ |I^6| &\leq c_m\Big(\int_0^d r^2(1+|\ln r|)^2(h^2\rho^{-2} + h^4(1+|\ln\rho|)^2) r\,dr\Big)^{1/2}|||z;\Omega(h)||| \\ &\leq c_m h^2(1+|\ln h|)\mathcal{N}. \end{aligned} \tag{3.21}$$

Here, expressions (2.19), (2.21) and (2.31) of boundary layer terms are taken into account, as well as linear dependence on $\ln h$ of the right-hand side (2.41) in problem (2.39), (2.40) for the term v^2 (compare to (2.38)). Moreover, d is the diameter of the support of cut-off function χ.

The remaining terms require some additional work. In view of relations (1.8) and (2.38)-(2.41) we have

$$\begin{aligned} I^1 &= (\partial_{n^h} v_m^0, z)_{\partial\omega_h\cap\Gamma(h)}, \\ I^2 &= I_1^2 + I_2^2 + I_3^2 := \\ &(\partial_{n^h} v_m^2, z)_{\partial\omega_h\cup\Gamma(h)} + (\Delta_x\chi(t_m^1 + t_m^2), z)_{\Omega(h)} + \lambda_m^0(\chi(t_m^1+t_m^2), z)_{\Omega(h)}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} |I_1^2| &\leq c_m\|\partial_{n^h} v_m^2; L_2(\partial\omega_h\cap\Gamma(h))\|\,\|z; L_2(\partial w_h\cap\Gamma(h))\| \\ &\leq c_m h^{1/2}(1+|\ln h|)h^{1/2}(1+|\ln h|)^{1/2}\mathcal{N}. \end{aligned} \tag{3.23}$$

In (3.23) it is taken into account the trace inequality (3.19), and the estimates

$$|\nabla_x^p v_m^2(x, \ln h)| \leq c_p r^{1-p}(1+|\ln h|), \quad p = 1, 2, \dots$$

for the solution of problem (2.39), (2.40) which follow from the theory of elliptic boundary problems in the domains with corners or conical points (see the paper [30], and also e.g., [36]) and from the analysis (2.43) of the right-hand side of equation (2.39).

By Remark 2.2 and formulae (2.19), (2.21), (2.31) and (2.37), (2.38) the estimates are obtained

$$\begin{aligned} |\widetilde{w}_m^1(\xi)| &= |w_m^1(\xi) - t_m^1(\xi)| \leq c\rho^{-1}(1+\rho)^{-1}, \\ |\widetilde{w}_m^2(\xi)| &= |w_m^2(\xi) - t_m^2(\xi)| \leq c(1+|\ln\rho|)(1+\rho)^{-1}, \end{aligned} \tag{3.24}$$

which means that

$$|I^5 - h^2 I_3^2| = \left|\lambda_m^0 \left(\chi(h\widetilde{w}^1 + h^2\widetilde{w}^2), z\right)_{\Omega(h)}\right|$$

$$\le ch\Big(\int\limits_0^d r^2(1+|\ln r|)(1+\rho)^{-2}(\rho^{-2} + h^2(1+|\ln\rho|)^2)rdr\Big)^{1/2} \tag{3.25}$$

$$\le ch^3(1+|\ln h|^{5/2})\mathcal{N}.$$

We continue the transformations

$$\begin{aligned} I^5 = I_1^5 + I_2^5 &:= \left(\nabla_x(hw_m^1 + h^2w_m^2), \nabla_x\chi z\right)_{\Omega(h)} - \left([\Delta_x, \chi](hw_m^1 + h^2w_m^2), z\right)_{\Omega(h)} \\ I_2^2 = I_4^2 + I_5^2 &:= \left(\chi\Delta_x(t_m^1 + t_m^2), z\right)_{\Omega(h)} + \left([\Delta_x, \chi](t_m^1 + t_m^2), z\right)_{\Omega(h)} . \end{aligned} \tag{3.26}$$

Here $[\Delta_x, \chi] = 2\nabla_x\chi \cdot \nabla_x + (\Delta_x\chi)$ is the commutator of Laplace operator with the cut-off function χ. The supports of coefficients of first order differential operator $[\Delta_x, \chi]$ are contained in the set $supp|\nabla_x\chi|$ which is located at the distance d_χ from the origin. Thus, taking into account relation (2.37) and Remark 2.2, we find

$$|I_2^5 - h^2 I_5^2| = ([\Delta_x, \chi](h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2), z)_{\Omega(h)} \tag{3.27}$$

$$\le c_m\Big(\int\limits_{d_\chi}^d \left(h^2\rho^{-4} + h^4\rho^{-2}\right)\Big|_{\rho=r/h} rdr\Big)^{1/2} \|z; L_2(\Omega(h))\| \le c_m h^3\mathcal{N}.$$

Moreover,

$$I_1^5 - h^2 I_4^2 = I_3^5 + I_4^5 := \left(\nabla_x(h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2), \chi z\right)_{\Omega(h)} - \left(\partial_{n^h}(hw_m^1 + h^2w_m^2), z\right)_{\partial\omega_h\cap\Omega(h)} . \tag{3.28}$$

Remark 3.3. The presence of corners on the boundary of domain Ξ may result in the singularities of derivatives of the boundary layer terms, therefore the inclusions $\chi\Delta_x\widetilde{w}_m^q \in L_2(\Omega(h))$ and $\chi\partial_{n^h}w_m^q \in L_2(\Gamma(h))$, in general are not valid. However, the terms in (3.28) may be well defined in the sense of duality obtained by the extension of scalar products $(\cdot,\cdot)_{\Omega(h)}$ and $(\cdot,\cdot)_{\Gamma(h)}$ in the Lebesgue spaces to the appropriate weighted Kondriatiev classes (see [17] and e.g., [36, Ch. 2]). Additional weighted factors are local, i.e., the factors are written in fast variables. That is why the norms of test functions z can be bounded as before by the constant $\mathcal{N}$. We point out that the method proposed below which involves the weighted norms can be avoided. The other possibility for the term I_1^5 is to rewrite the gradient ∇_x in curvilinear coordinates n, s, pass to the fast variables and take into account the integral identities for problems (2.17) and (2.26) with the test function $\xi \longmapsto \chi(x)z(x)$. ■

By its definition, the function $\widetilde{w}_m^1$ remains harmonic, and $\widetilde{w}_m^2$ verifies the equation

$$\Delta_\xi\widetilde{w}_m^2(\xi) = -L^1(\xi_1, \nabla_\xi)\widetilde{w}_m^1(\xi), \quad \xi \in \Xi; \tag{3.29}$$

here are taken into account splittings (2.13) and (2.42) of the Laplace operator. Therefore,

$$\Delta_x(h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2) = h^2L^1\widetilde{w}_m^2 + L^2(h\widetilde{w}^1 + h^2\widetilde{w}^2). \tag{3.30}$$

In (3.30) the operators L^q are written in the slow variables and the function $\widetilde{w}^q$ in fast variables (in contrast to (3.29) where $\Delta_\xi = h^{-2}L^0(\partial_n, \partial_s)$ and $L^1(\xi_1, \nabla_\xi) = h^{-1}L^1(n, \partial_n, \partial_s)$

). Owing to (3.30), (2.37) in application of Remark 3.3, it follows that

$$|I_3^5| \leq c_m \mathcal{N}\Big(\sum_{q=0}^{2} h^{-2q} \int_0^d r^2(1+|\ln r|)^2 (r^2 h^4 \rho^{-2(1+\rho)}+ \tag{3.31}$$

$$r^4(h^2\rho^{-2(2+\rho)} + h^4\rho^{-2(1+\rho)}))|_{\rho=r/h} r dr\Big)^{-1/2} \leq c_m h^3 \mathcal{N}.$$

It suffices to process the difference of integrals from (3.22) and (3.28):

$$I^1 - I_4^5 = -\left(\partial_{n^h}(hw_m^1 + h^2 w_m^2 - v_m^0), z\right)_{\partial\omega_h \cap \Gamma(h)}.$$

Using the same arguments, already applied to derivation of boundary conditions in problems (2.17) and (2.26), we refer to formulae (2.14)-(2.13) and (2.28), and obtain that

$$|I^1 - I_4^5| \leq c_m h^{1/2}(1+|\ln h|)^{1/2} \mathcal{N} h^2 (mes_1 \partial\omega_h)^{1/2} \leq c_m h^3 (1+|\ln h|)^{1/2}\mathcal{N}. \tag{3.32}$$

Collecting estimates (3.21), (3.23), (3.25), (3.27), (3.31), (3.32) of the terms in (3.20), we arrive at the following estimate of value (3.18)

$$\tau \leq c_m h^3 (1+|\ln h|)^{5/2}. \tag{3.33}$$

We point out, that the maximal exponent $5/2$ of the logarithmic factor is inherited from estimate (3.25). We are ready now to verify the theorem about the asymptotics, which is the main result of this section

Theorem 3.4. *For any positive eigenvalue λ_m^0 of multiplicity $\varkappa_m$ for problem (1.8), see (2.48), there exist numbers $\mathbf{c}_m > 0$ and $h_m > 0$ such that for $h \in (0, h_m]$ the eigenvalues $\lambda_m^h, \dots, \lambda_{m+\varkappa_m-1}^h$ for problem (1.3), (1.4) and except for all other eigenvalues in sequence (1.5) satisfy the following inequalities*

$$|\lambda_q^h - \lambda_m^0 - h^2\lambda^{q\prime}| \leq \mathbf{c}_m h^3 (1+|\ln h|)^{5/2}, \quad q = m, \dots, m+\varkappa_m - 1. \tag{3.34}$$

Moreover, there is a constant $\mathbf{C}_m$ and columns $a^{hm}, \dots, a^{hm+\varkappa_m-1}$ which define an unitary matrix of the size $\varkappa_m \times \varkappa_m$ such that

$$\|v^{q0} + \chi(hw^{q1} + h^2 w^{q2}) + h^2 v^{q2} - \sum_{p=m}^{m+\varkappa_m-1} a_p^{hq} u_p^h; H^1(\Omega(h))\| \leq \mathbf{C}_m h (1+|\ln h|)^{5/2}, \tag{3.35}$$

$$q = m, \dots, m+\varkappa_m - 1.$$

By v^{q0} is denoted linear combination (2.49) of eigenfunctions in problem (1.8), constructed in the end of Section 2.4, and w^{q1}, w^{a2} and v^{q2} are given functions which are determined for fixed v^{q0} in the way described in §2, finally $\lambda^{q\prime}$ is an eigenvalue of the matrix $\mathbf{M}$ with coefficients (2.53). In the case of a simple eigenvalue λ_m^0 (i.e., $\varkappa_m = 1$), we have $v^{m0} = v_m^0$ the corresponding eigenfunction, and $\lambda^{m\prime} = \lambda_m'$ is given by (2.47).

Proof. Given eigenvectors $a^m, \dots, a^{m+\varkappa_m-1}$ of the matrix $\mathbf{M}$, we construct linear combinations (2.49) and the associated appropriate terms in asymptotic ansatz (1.11). As a result, approximation solutions $\left\{(\lambda_q^0 + h^2\lambda^{q\prime})^{-1}, U^q\right\}$ for $q = m, \dots, m+\varkappa_m - 1$ are obtained for the abstract spectral problem (3.16).

Let $\lambda^{q\prime}$ be an eigenvalue of the matrix $\mathbf{M}$ of multiplicity κ_q, i.e.,

$$\lambda^{q-1\prime} < \lambda^{q\prime} = \cdots = \lambda^{q+\kappa_q-1\prime} < \lambda^{q+\kappa_q\prime}. \tag{3.36}$$

We choose the factor c_* in the value $\tau_* = c_* h^2$ in Lemma 3.2 so small that the segment

$$[(\lambda_m^0 + h^2\lambda^{q\prime})^{-1} - c_* h^2, (\lambda_m^0 + h^2\lambda^{q\prime})^{-1} + c_* h^2] \tag{3.37}$$

does not contain the approximation eigenvalues $(\lambda_m^0 + h^2\lambda^{p\prime})^{-1}$ with $p \notin \{q, q+\kappa_q - 1\}$. Then Lemma 3.2 delivers the eigenvallues $\mu^h_{i(q)}, \dots, \mu^h_{i(q+\kappa_q-1)}$ of the operator K^h such that

$$(3.38) \qquad |\mu^h_{i(q)} - (\lambda_m^0 + h^2\lambda^{q\prime})^{-1}| \le \tau \le \mathbf{c}_m h^3(1+|\ln h|)^{5/2}, \quad p = q, \dots, q+\kappa_q - 1\,.$$

We here emphasize that, at the time being, we cannot infer that these eigenvalues are differents. At the same moment, the second part of Lemma 3.2 gives the normed colums $b^{hp} = (b^{hp}_{k_{mq}}, \dots, b^{hp}_{k_{mq}+N_{mq}-1})$ veryfying the inequalities

$$(3.39) \qquad \|U^p - \sum_{k=k_{mq}}^{k_{mq}+N_{mq}-1} b_k^{hp} u_k^h; H^1(\Omega(h))\| \le c\frac{\tau}{\tau_*} \le ch(1+|\ln h|)^{5/2}.$$

Here $\{\mu^h_{k_{mq}}, \dots, \mu^h_{k_{mq}+N_{mq}-1}\}$ implies the list of all eigenvalues of the operator K^h in segment (3.37). Note that the numbers k_{mq} and N_{mq} can depend on the parameter h but this fact is not reflected in the notation. Since

$$(3.40) \qquad \begin{gathered} \|h\chi w^1; H^1(\Omega(h))\| \le ch, \quad \|h^2\chi w^2; H^1(\Omega(h))\| \le ch^2(1+|\ln h|), \\ \|h^2 v^2; H^1(\Omega(h))\| \le ch^2(1+|\ln h|)^2, \end{gathered}$$

the normalistion condition (1.9) for the eigenfunctions of problem (1.8) and similar conditions for eigenvectors of the matrix $\mathbf{M}$ ensure that

$$(3.41) \qquad |(U^p, U^t)_{L_2(\Omega(h))} - \delta_{p,t}| \le ch, \quad p, t = q, \dots, q+\kappa_q + 1.$$

On the other hand, inequalities (3.39) and the orthogonality and normalisation conditions (1.6) for eigenfunctions u_k^h of problem (1.3), (1.4) lead to the relation

$$(3.42) \qquad \Big|(U^p, U^t)_{L_2(\Omega(h))} - \sum_{k=k_{mq}}^{k_{mq}+N_{mq}-1} b_k^{hp} b_k^{ht}\Big| \le ch(1+|\ln h|)^{5/2}.$$

Formulas (3.41) and (3.42) are true simultaneously if and only if

$$(3.43) \qquad N_{mq} \ge \kappa_q\,.$$

To prove that actually the sign = occurs in (3.43), we first of all, notice that, for a sufficiently small $h > 0$, the relations of type (3.43) are valid for all eigenvaliues $\lambda_1^0, \dots, \lambda_m^0$ of problem (1.8) and all eigenvalues $\lambda^{q\prime}$ of the associated matrices $\mathbf{M}$. We have verified above Proposition 3.1 that each eigenvalue λ_p^h and the corresponding eigenfunction u_p^h of singularly perturbed problem (1.3), (1.4) converge to an eigenvalue and an eigenfunction of the limit problem (1.8), respectively. This observation ensures that the number of entries of the eigenvalue sequence (1.5), which live on the interval $(0, \lambda_m^0)$, does not exceed $m+\varkappa_m-1$ for a small $h > 0$. Summing up the inequalities (3.43) over all $\lambda_1^0, \dots, \lambda_m^0$ and $\lambda^{q\prime}$, we conclude that the equalities $N_{mq} = \kappa_q$ are necessary. Moreover, we now are able to confirm that the eigenvalues $\mu^h_{i(q)}, \dots, \mu^h_{i(q+\kappa_q-1)}$ can be chosen different one from another. Indeed, we take $\tau_* = C_* h^3(1+|\ln h|)^{5/2}$ in Lemma 3.2 and fix C_* so large that the inequality (3.39) with the new bound c/C_* still warrants that the segment

$$(3.44) \qquad \Upsilon_q(h) = \Big[(\lambda_m^0 + h^2\lambda^{q\prime})^{-1} - C_* h^3(1+|\ln h|)^{5/2}, (\lambda_m^0 + h^2\lambda^{q\prime})^{-1} + C_* h^3(1+|\ln h|)^{5/2}\Big]$$

contains exactly κ_q eigenvalues of the operator K^h. It suffices to mention two facts. First, for a small $h > 0$, the intervals $\Upsilon_q(h)$ and $\Upsilon_p(h)$ with $\lambda^{q\prime} \ne \lambda^{p\prime}$ do not intersect. Second, any eigenvalue $\mu_k^h = (\lambda_k^h)^{-1}$ in the interval (3.44) meets the inequality (3.34). ■

Remark 3.5. Estimates (3.40) show that the bound in (3.35) is large than norms of the terms w^{q1}, w^{q2} and v^{q2} included into the approximation solution and, therefore, the estimate (3.35) remaines valid for the function v^{q0} alone, without three correcting terms. This is the usual situation in the asymptotic analysis of singular spectral problems: One needs to construct additional asymptotic terms of eigenfunctions in order to prove that the correcting term in the asymptotics of an eigenvalue is found correctly. In principal, one can employ the general procedure [28] and construct higher asymptotic terms of eigenvalues and eigenfunctions. We keep the boundary layer and regular corrections in the estimate (3.35) because they form so-called asymptotic *conglometate* which is replicated in the asymptotic series (see [28] and [35]; actually the notion of asymptotic conglomerates was introduced in [35]).

4. Other geometrical forms and boundary conditions

4.1. **Perturbation of a domain with the Dirichlet boundary conditions.** Let us consider the spectral problem with equation (1.3) and the Dirichlet boundary conditions

$$u^h(x) = 0, \quad x \in \Gamma(h), \tag{4.1}$$

or the mixed boundary conditions

$$u^h(x) = 0, \quad x \in \Gamma(h) \setminus \partial\omega_h, \quad \partial_{n^h} u^h(x) = 0, \quad x \in \partial\omega_h \cap \Gamma(h). \tag{4.2}$$

Eigenvalues for these two spectral problem form a sequence, of the same form as in (1.5), given by

$$0 < \lambda_1^h < \lambda_2^h \leq \lambda_3^h \leq \cdots \leq \lambda_m^h \leq\rightarrow +\infty \tag{4.3}$$

with the corresponding eigenfunction subject to condition (1.6). The notation for attributes of three spectral problems (1.3), (1.4) and (1.3), (4.1) or (1.3), (4.2) is the same, without any misunderstanding. The peculiarity of spectral problems introduced in this section is the absence of the eigenvalue $\lambda_0^h = 0$, compare (4.3) with (1.5). The asymptotic ansatz given by (1.10) and (1.11) keep their validity and the first terms λ_m^0, v_m^0 are given by the solutions of Dirichlet spectral problem

$$\Delta_x v^0(x) = \lambda^0 v^0(x), \quad x \in \Omega, \quad v^0(x) = 0, \quad x \in \Gamma, \tag{4.4}$$

which admits the infinite sequence of eigenvalues

$$0 < \lambda_1^0 < \lambda_2^0 \leq \lambda_3^0 \leq \cdots \leq \lambda_m^0 \leq\rightarrow +\infty, \tag{4.5}$$

compare again with (1.7), and the corresponding eigenfunctions $v_1^0, v_2^0, v_3^0, \ldots, v_m^0, \ldots$ are subject to the orthogonality and normalisation conditions (1.9). The construction of asymptotics for the Dirichlet boundary conditions on the non perturbed part of the contour $\Gamma(h)$ is much simpler compared to the case of Neumann conditions. In particular, the second term w^2 of boundary layer type can be neglected, and we can restrict ourselves to analysis of problems for w^1 and v^2. Indeed, the Dirichlet boundary conditions on non compact part of the boundary $\partial\Xi$ turn out to provide the decay at infinity of both the functions w^1 and w^2, instead of the case of Neumann conditions. Hence, w^1 and w^2 in the present case are of the boundary layer type. It means that the decomposition for $|\xi| \to \infty$ at the term w^2 is free from the non-decaying parts (compare to formulae (2.31)), therefore there is no discrepancy from the term w^2 in the problem for v^0.

Assume that, in the same way as in section 2.1, $\lambda^0 = \lambda_m^0$ is a simple eigenvalue in problem (4.4) and $v^0 = v_m^0$ is the corresponding eigenfunction, in particular $\|v^0; L_2(\Omega)\| =$

1. In the vicinity of the point O we have

$$v^0(x) = n\partial_n v^0(O) + \frac{1}{2}n^2\partial_n^2 v^0(O) + ns\partial_n\partial_s v^0(O) + O(r^3) \tag{4.6}$$
$$= h\xi_1\partial_n v^0(O) + h^2\left(\frac{1}{2}\xi_1^2\partial_n^2 v^0(O) + \xi_1\xi_2\partial_n\partial_s v^0(O)\right) + O(h^3).$$

Thus, the principal term of the boundary layer type should be given by the solution to boundary value problem

$$-\Delta_\xi w^1(\xi) = 0, \quad \xi \in \Xi, \quad w^1(\xi) = 0, \ \xi \in \partial\Xi \setminus \partial\omega, \tag{4.7}$$

with the boundary conditions

$$w^1(\xi) = -\xi_1\partial_n v^0(O), \quad \xi \in \partial\omega \cap \partial\Xi, \tag{4.8}$$

in the case of (4.1), or with the conditions

$$\partial_\nu w^1(\xi) = -\nu_1(\xi)\partial_n v^0(O), \quad \xi \in \partial\omega \cap \partial\Xi, \tag{4.9}$$

for mixed boundary value problem (1.3), (4.2). In the two cases, for the solution w^1 the following relations are fulfilled

$$w^1(\xi) = W(\xi)\partial_n v^0(O), \tag{4.10}$$

$$W(\xi) = \frac{\mathbf{m}}{\pi}\frac{\xi_1}{\rho^2} + O(\rho^{-2}) = -\frac{\mathbf{m}}{\pi}\rho^{-1}\cos\varphi + O(\rho^{-2}) \tag{4.11}$$

where the decomposition (4.11) is described already in Remark 2.2. Let us note that $\Xi \subset \mathbb{R}^2_-$; hence the asymptotic term detached in (4.11) is negative.

Lemma 4.1.
- *For the mixed boundary value problem (4.7), (4.9), the constant* **m** *in decomposition (4.11) is given by (2.24).*
- *For the Dirichlet problem (4.7), (4.8) the constant* **m** *in decomposition (4.11) is given by formula*

$$\mathbf{m}(\Xi) := \mathbf{m} = -\int_\Xi |\nabla_\xi W(\xi)|^2 d\xi - mes_2\omega. \tag{4.12}$$

Proof. The sum $Y(\xi) = \xi_1 + W(\xi)$ turns out to be a solution of problem with the homogeneous boundary conditions on $\partial\omega \cap \partial\Xi$. Therefore, for the mixed boundary value problem, in the same way as in (2.22) and (2.23), we obtain the following relations

$$\int_\Xi |\nabla_\xi W(\xi)|^2 d\xi + mes_2\omega = \int_{\partial\omega\cap\partial\Xi} Y\partial_\nu W ds_\xi = \int_{\partial\omega\cap\partial\Xi} (Y\partial_\nu W - W\partial_\nu Y) ds_\xi \tag{4.13}$$
$$= \lim_{R\to\infty} \int_{\{\xi\in\mathbb{R}^2_-:\rho=R\}} \left(W\partial_\rho Y - Y\partial_\rho W\right) ds_\xi$$
$$= \frac{\mathbf{m}}{\pi}\int_{-\pi/2}^{\pi/2} \left(\rho^{-1}\cos^2\varphi + \rho\cos\varphi\rho^{-2}\cos\varphi\right)\rho|_{\rho=R} d\varphi + O(R^{-1})$$
$$= \mathbf{m} + O(R^{-1}).$$

For the Dirichlet problem, the first equality in (4.13) is replaced by the formula

$$-\int_\Xi |\nabla_\xi W(\xi)|^2 d\xi - mes_2\omega = -\int_{\partial\omega\cap\partial\Xi} W\partial_\nu Y ds_\xi.$$

■

The problem for regular term v^2 is defined in exactly same way as problem (2.39), (2.40), however the reasons mentioned above lead to the following form of the right-hand side f^2

$$f^2(x) = \lambda^0\chi(x)t^1(n,s) + \Delta(\chi(x)t^1(n,s)), \quad t^1(n,s) = \pi^{-1}m\partial_n v^0(O)n(n^2+s^2)^{-1}. \tag{4.14}$$

compare to formula (2.41) where in addition the term t^2 comes from decomposition (2.37), $q = 2$ which is null in the present case. Therefore the function v^2 can be determined by a solution of equation (2.39) with the boundary conditions

$$v^2(x) = 0, \quad x \in \Gamma. \tag{4.15}$$

Let us observe that it is not required that the cut-off function verifies some additional conditions: in any case the Dirichlet conditions turn out to be homogeneous. Since we are going to provide the term w^2 with a decay at infinity, the right-hand side (2.27) of the Poisson equation in problem (2.26) is modified and the equation takes the form

$$-\Delta_\xi w^2(\xi) = \varkappa(0)\left(\partial_{\xi_1}\widetilde{w}^1(\xi) - 2\xi_1\partial^2_{\xi_2}\widetilde{w}^1(\xi)\right), \quad \xi \in \Xi, \tag{4.16}$$

where $\widetilde{w}^1(\xi) = w^1(\xi) - X(\xi)t^1(\xi)$, $X \in C^\infty(\mathbb{R}^2)$ is a cut-off function, $X = 1$ for $\rho > 2R_0$ and $X = 0$ for $\rho < R_0$, and t^1 is the principal term of asymptotics for w^1_0. In the case (4.1) the boundary conditions for w^2 are given as follows

$$w^2(\xi) = 0, \quad \xi \in \partial\Xi \setminus \partial\omega, \tag{4.17}$$

$$w^2(\xi) = -\frac{1}{2}\xi_1^2\partial_n^2 v^0(O) - \xi_1\xi_2\partial_n^2\partial_s v^0(O), \quad \xi \in \partial\omega \cap \partial\Xi, \tag{4.18}$$

and in the case (4.2) formula (4.18) is replaced by the following one

$$w^2(\xi) = G^2(\xi), \quad \xi \in \partial\omega \cap \partial\Xi; \tag{4.19}$$

moreover the term $G^2_1 = 0$ and G^2_2 in formula (2.28) are not changed and in accordance with the decomposition (4.6) we have

$$G^2_3(\xi) = -\xi_1\nu_1(\xi)\partial_n^2 v^0(O) - (\xi_1\nu_2(\xi) + \xi_2\nu_1(\xi))\partial_n\partial_s v^0(O).$$

Due to the Dirichlet conditions (4.17) on non compact part of the boundary two problems (4.16)-(4.18) and (4.16), (4.17), (4.19) admit decaying solutions. In view of Remark 2.2 it is straightforward to verify the estimates

$$|\nabla^p_\xi w^2(\xi)| \le c_p\rho^{-1-p}, \quad \rho \ge R_0, \quad p = 0,1,2,\dots$$

Hence, the function w^2 looses nondecreasing asymptotic term t^2, in the framework of rearrangement of discrepancies [25], [28]; the resulting from t^2 term are of order $O(\rho^{-2})$ and in formula (2.29) are transfered on the right-hand side of equation (2.39) (compare (4.14) with (2.41), (2.38)). Repetition with obvious changes of arguments from Section 2.3 shows that the compatitbility condition for problem (2.39), (4.15) in the class of bounded functions is equivalent to formula

$$\lambda' = \mathbf{m}|\partial_n v^0(O)|^2. \tag{4.20}$$

In the case of multiple eigenvalue λ^0_m in problem (4.4) with condition (2.48) satisfied, the members of asymptotic ansatz are determined exactly in the same way, as in the end of Section 2.4. The only exception is formula (2.53), which now reads as follows

$$\mathbf{M}_{jk} = \mathbf{m}(\Xi)\partial_n v^0_{m+k}(O)\partial_n v^0_{m+j}(O); \tag{4.21}$$

where $\mathbf{m} = \mathbf{m}(\Xi)$ is the integral attribute of the domain Ξ, described in Lemma 4.1.

Remark 4.1. We leave the denotation W for odd extension from the domain Ξ onto the domain Ξ^{00} of the function W, which appears in (4.10) and (4.11), (see (2.25) and Figure 3). By virtue of the homogeneous Dirichlet boundary conditions on $\partial\Xi \setminus \partial\omega$, it is harmonic function in Ξ^{00}, subject to the condition

$$W(\xi) = -\xi_1, \quad \xi \in \partial\Xi^{00},$$

in the case (4.8) and to the condition

$$\partial_\nu(\xi) = -\nu_1(\xi), \quad \xi \in \partial\Xi^{00}.$$

in the case (4.9).This type of function is employed in [41] for the description of polarisation tensors and of virtual mass, respectively. By Lemma 4.1 the quantity $\mathbf{m}(\Xi)$ from decomposition (4.11) is twice the upper left-hand element in the matrix associated with the polarisation tensor. ■

The justification of obtained formal asymptotics practically repeats arguments already presented in section 2.4. The only exception is the fact, that the bounded solution v^0 of problem (2.39), (4.15) can be represented in the from $v^2(x) = \Phi(\varphi) + O(r)$, therefore is not any element of the Sobolev space $H^1(\Omega)$. Thus, when constructing the global asymptotic approximation for the eigenfunction u_m^h, the term v^2 is multiplied by the cut-off function X_h which equals to $X(h^{-1}n, h^{-1}s)$ in the vicinity of the point O and to one on the remaining part of the domain Ω. We point out, that additional discrepancy resulting from the term v^2, cancels in the principal part with the discrepancy resulting from the multiplication of the asymptotic term t^1 by the cut-off function $X(\xi)$ on the right-hand side of equation (4.16). Finally, we formulate the result.

Theorem 4.2. *For any eigenvalue λ_m^0 in problem (4.4) of multiplicity $\varkappa_m$ (see (2.48)) there exist the constants $\mathbf{c}_m > 0$ and $h_m > 0$ such that for $h \in (0, h_m]$ the eigenvalues $\lambda_m^h, \dots, \lambda_{m+\varkappa_m-1}^h$ in the Dirichlet problem (1.3), (4.1) (respectively, in the mixed boundary value problem (1.3), (4.2)), but not all other members of sequence (4.3), satisfy the following inequalities*

$$|\lambda_q^h - \lambda_m^0 - h^2\lambda^{q\prime}| \le \mathbf{c}_m h^3, \quad q = m, \dots, m + \varkappa_m - 1. \tag{4.22}$$

Moreover, there exist the constant $\mathbf{C}_m$ and the columns $a^{hm}, \dots, a^{hm+\varkappa_m-1} \in \mathbb{R}^{\varkappa_m}$ which form an unitary matrix of dimension $\varkappa_m \times \varkappa_m$, such that

$$\left\| v^{q0} + \chi h w^{q1} + X_h v^{q2} - \sum_{p=m}^{m+\varkappa_m-1} a_p^{hq} u_p^h; H^1(\Omega(h)) \right\| \le \mathbf{C}_m h, \tag{4.23}$$

In (4.23) v^{q0} stands for the linear combination (2.49) of the eigenfunctions in problem (4.4) subject to the orthogonality and normalisation conditions (1.9). The columns of coefficients a^{q0} in (2.49) satisfy condition (2.50). The quantities $\lambda^{q\prime}$ in (4.22) are given by eigenvalues, and the columns a^{q0} by eigenvectors, of the $(\varkappa_m \times \varkappa_m)$-matrix $\mathbf{M}$ with the coefficients (4.21). The terms w^{q1} and w^{q2} of the boundary layer type are determined by the function v^{q0} while solving problems (4.7), (4.8) and (4.16)-(4.18) ((4.7), (4.9) and (4.16), (4.17), (4.19) in the case of mixed boundary conditions), respectively. The function v^{q2} is a bounded solution of problem (2.39), (4.15) with the right-hand side (4.14), such a solution exists provided $\varkappa_m$ compatibility conditions in form (2.52) are verified. In the particular case of a simple eigenvalue λ_m^0, it follows that $v^{m0} = v_m^0$ is the coresponding normalised eigenfunction in problem (4.4), and the unique compatibility condition of problem (2.39), (4.15) furnishes the quantity $\lambda_m' = \lambda^{m\prime}$ from formula (4.20). ■

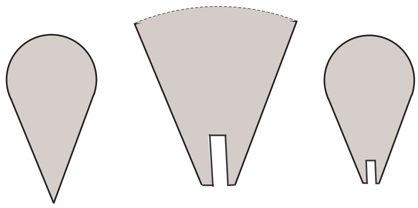

FIGURE 5. Perturbation near the corner point.

The absence of logarithms in majorants (4.22) and (4.23) (compare with (3.34) and (3.35) in Theorem 4.2) can be explained in the following way. First, there is no singular term t^2 with the logarithm (see (2.31) and (2.37)), furthermore the functions $u \in \mathring{H}^1(\Omega(h))$ satisfy inequality (3.1), where in the middle the weight factor r^{-1} stands instead of $r^{-1}(1 + \ln|r|)^{-1}$ The verification of such an inequality is performed by means of the Friedrichs inequality on the arc $(-\pi/2, \pi/2) \ni \varphi$ and for the Dirichlet boundary conditions on $\Gamma(h) \setminus \partial\omega_h$, see e.g., [36, Ch. 2] for the details.

Remark 4.3. Since the boundary layer terms w^{q1} and w^{q2} have similar behaviour at infinity, we have excluded the second one from the approximation solution in (4.23) in accordance with the concept of asymptotic conglomerates [28], [35]. ■

4.2. **Perturbation of the boundary in the vicinity of a corner point.** Assume that in the vicinity of the origin the domain Ω coincides with the angle $\mathbb{K} = \{x : r > 0, |\varphi| < \alpha/2\}$ where $\alpha \in (0, 2\pi]$ is the opening of the angle, (r, φ) are the polar coordinates, and we set $x_1 = -r\cos\varphi$ and $x_2 = r\sin\varphi$. Given a domain ω with the origin O in its interior, we denote

$$\omega_h = \left\{x : \xi := h^{-1}x \in \omega\right\}. \tag{4.24}$$

In the domain $\Omega(h) = \Omega \setminus \overline{\omega}_h$, see Figure 5, with the piece-wise smooth boundary $\Gamma(h) = \partial\Omega(h)$, we consider equation (1.3) along with the boundary conditions (4.1) or (4.2). In contrary to the previous sections, the stretching of coordinates is performed for the Cartesian coordinate system (compare formula (4.24) to (1.1)) and, therefore, the derivation of asymptotics becomes now much simpler. We restrict our analysis to a simple eigenvalue λ_m^0 of the limit Dirichlet problem (4.4) and the corresponding eigenfunction v_m^0 normalised by relation (1.9). In the vicinity of the corner point O the function v_m^0 admits the decomposition

$$v_m^0(x) = K_m r^{\pi/2} \cos\left(\frac{\pi}{\alpha}\varphi\right) + O(r^{\min\{2\pi/\alpha, 2+\pi/\alpha\}}), \quad r \to 0, \tag{4.25}$$

where K_m is a constant (it is the so-called intensity factor). Such a form of singular function can be found, e.g. by an application of the Fourier method, and we refer the reader for all the details of the derivation, e.g., to [36, Ch. 2]. By the formula $v_m^0(x) = K_1 h^{\pi/\alpha}\rho^{\pi/\alpha}\cos(\pi\alpha^{-1}\varphi) + o(h^{\pi/2})$ on $\partial\Omega(h) \setminus \partial\Omega$, it follows that the decomposition of boundary layer type starts with the term $h^{\pi/\alpha}\omega_m^{\pi/\alpha}(\xi)$, which compensates the principal part of the discrepancy in the boundary conditions and is given by a solution of the boundary value problem

$$-\Delta_\xi w_m^{\pi/\alpha}(\xi) = 0, \quad \xi \in \Xi := \mathbb{K} \setminus \overline{\omega}, \quad w_m^{\pi/\alpha}(\xi) = 0, \quad \xi \in \partial\Xi \setminus \partial\omega, \tag{4.26}$$

$$w_m^{\pi/\alpha}(\xi) = -K_m \rho^{\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right), \quad \xi \in \partial\omega \cap \partial\Xi, \tag{4.27}$$

or

$$\frac{\partial w_m^{\pi/\alpha}}{\partial \nu}(\xi) = -K_m \frac{\partial}{\partial \nu} \rho^{\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right), \quad \xi \in \partial\omega \cap \partial\Xi, \tag{4.28}$$

where (4.28) is given for (4.2) and (4.27) is the condition for (4.1). Since the homogeneous Dirichlet conditions are prescribed on the non compact part $\partial\Xi$ of the boundary, problem (4.26), (4.27) admits a unique solution $w_m^{\pi/\alpha}$ with the decay at infinity. The following formulae are valid,

$$w_m^{\pi/\alpha}(\xi) = K_m W(\xi), \tag{4.29}$$

$$W(\xi) = \frac{\mathbf{m}}{\pi} \rho^{-\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right) + O(\rho^{-2\pi/\alpha}), \quad \rho \to \infty. \tag{4.30}$$

Here W is a solution to problem (4.26), (4.27) for $K_m = 1$.

Remark 4.4. By a simple repetition of arguments given in the proof of Lemma 4.1, with the evident modifications if necessary, we arrive at the equality

$$m(\Xi) := m = \mp \int_\Xi |\nabla_\xi W(\xi)|^2 d\xi \mp \int_{\omega \cap \mathbb{K}} \left| \nabla_\xi \left(\rho^{\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right)\right)\right|^2 d\xi; \tag{4.31}$$

where the sign $-$ relates to the condition (4.1), and the sign + is given for the case of (4.2). Such formulae can be derived from relations (2.24) and (4.12) by means of a conformed mapping technique, with the appropriate transformation of the corner $\mathbb{K}$ onto the half-plane $\mathbb{R}^2_-$. Unfortunately, the conformal mapping is not applicable in a simple way to equation (1.3), the reason is the form of the right-hand side with the additional weight factor which makes the asymptotic constructions provided in section 6.1 much more involved. ■

The asymptotic ansatz for solutions of spectral problems (1.3), (4.1) and (1.3), (4.2) should be taken in the following form:

$$\lambda_m^h = \lambda_m^0 + h^{2\pi/\alpha} \lambda_m' + \dots \tag{4.32}$$

$$u_m^h(x) = v_m^0(x) + \chi(x) h^{\pi/\alpha} w_m^{\pi/\alpha}(\xi) + h^{2\pi/\alpha} v_m^{2\pi/\alpha}(x) + \dots \tag{4.33}$$

The correction term $v_m^{2\pi/\alpha}$ of regular type is given by a solution to problem (2.39), (4.15) with the right-hand side

$$f^{2\pi/\alpha}(x) = \lambda^0 \chi(x) t^{\pi/\alpha}(x) + \Delta_x(\chi(x) t^{\pi/\alpha}(x)) = \lambda^0 \chi(x) t^{\pi/\alpha}(x) + [\Delta_x, \chi(x)] t^{\pi/\alpha}(x),$$

$$t^{\pi/\alpha}(x) = K_1 \frac{\mathbf{m}}{\pi} r^{-\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right).$$

It is taken into account that $t^{\pi/\alpha}$ is a harmonic function. In view of $f^{2\pi/\alpha}(x) = O(r^{-\pi/\alpha})$ for $r \to 0$, solutions to problem (2.39), (4.15) should be found in the class of smooth functions on $\overline{\Omega} \setminus O$, such that

$$|\nabla_x^p v^{2\pi/\alpha}(x)| \le c_p r^{\min\{\pi/\alpha, 2-\pi/\alpha\}-p}, \quad p = 0, 1, \dots$$

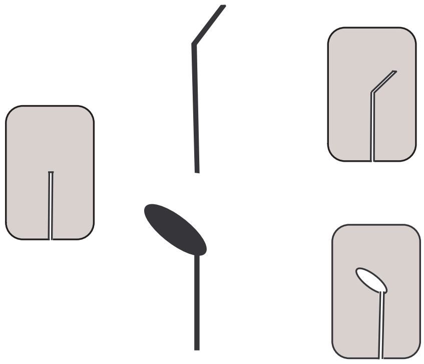

FIGURE 6. The magistral crack with a kink or cavity at the tip.

It is not difficult to see (we refer the reader to [36, Ch. 2], for the details) that the solution with such properties does exist if and only if the following compatibility condition is verified

$$\int_{\Omega} (\lambda'_m v_m^0 + f^{2\pi/\alpha}) v_m^0 dx = 0. \tag{4.34}$$

We recall here that λ_m^0 is a simple eigenvalue. By an application of the Green formula in the domain $\Omega \setminus \mathbb{B}_\rho$ in the same way as before, we can transform the compatibility condition (4.34) into the formula for the asymptotic correction in the ansatz (4.32) for the eigenvalue

$$\lambda'_m = \mathbf{m} K_m^2. \tag{4.35}$$

We are not going to analyse the case of multiple eigenvalues. Since the theorem on asymptotics and its proof can be presented in exactly the same way as before, no new ideas is required. For this reason, we are going to derive several asymptotic terms in the most interesting case of the Neumann problem (1.3), (1.4) for domains with cracks, see Figure 6

4.3. **Spectral problems for domains with cracks.** Let $\alpha = 2\pi$ and let λ_m^0 be a simple eigenvalue for the limit Neumann problem (1.8). The eigenfunction v_m^0, normalised in the space $L_2(\Omega)$, can be decomposed as follows

$$\begin{aligned} v_m^0(x) = v_m^0(\mathcal{O}) + K^1 r^{1/2} \cos\frac{\varphi}{2} + K^2 r \cos\varphi + K^3 r^{3/2} \sin\Big(\frac{3}{2}\varphi\Big) \\ + K^4 r^2 \cos 2\varphi - \frac{1}{4}\lambda_m^0 v_m^0(\mathcal{O}) r^2 + O(r^{5/2}), \quad \tau \to 0. \end{aligned} \tag{4.36}$$

The four terms in the right-hand side, witch contain the intensity factor K_m^p are denoted by $K_m^p r^{p/2} \Phi_p(\varphi)$, $p = 1, 2, 3, 4$. The last asymptotic term comes out from the term $\lambda_m^0 v_m^0$ in the equation from problem (4.4) (compare with the procedure of asymptotic construction [17], which is retold e.g., in [36, Ch. 6]). Due to formula (4.36), the following asymptotic ansätze are proposed:

$$\lambda_m^h = \lambda_m^0 + h\lambda_m^{(2)} + h^{3/2}\lambda_m^{(3)} + h^2\lambda_m^{(4)} + \dots, \tag{4.37}$$

$$u_m^h(x) = v_m^0(x) + \sum_{j=2}^{4} h^{j/2} v_m^{(j)}(x) + \chi(x) \sum_{p=1}^{4} h^{p/2} w_m^{(p)}(\xi) + \dots \tag{4.38}$$

The terms w_m^1 and w_m^2 of the boundary layer type are given by solutions of the problems

$$-\Delta_\xi w_m^{(p)}(\xi) = 0, \quad \xi \in \Xi, \quad \partial_\nu w_m^{(p)}(\xi) = -K^p \partial_\nu (\rho^{p/2} \Phi_p(\varphi)), \quad \xi \in \partial\Xi. \tag{4.39}$$

The right-hand sides of the boundary conditions have compact supports and the integrals of the functions over $\partial\Xi$ vanish. This implies the existence of a unique solution to (4.39) which decay at infinity. In addition, the following representation $w_m^p = K_m^p W_m^p$ is useful, and

$$W^p(\xi) = \frac{1}{\pi} \sum_{q=1}^{3} \frac{1}{q} \mathbf{m}_{pq} \rho^{-q/2} \Phi_q(\varphi) + O(\rho^{-2}), \quad \rho \to \infty. \tag{4.40}$$

Lemma 4.2. *The matrix* $(\mathbf{m}_{pq})_{p,q=1}^2$ *of size* 2×2 *is symmetric and positive definite,*

$$\mathbf{m}_{pq}(\Xi) := \mathbf{m}_{pq} = (\nabla_\xi W^p, \nabla_\xi W^q)_\Xi + (\nabla_\xi(\rho^{p/2}\Phi_p), \nabla_\xi(\rho^{p/2}\Phi_q))_\omega \,. \tag{4.41}$$

Proof follows the proof of Lemma 4.1. The functions $Y^p(\xi) = \rho^{p/2}\Phi_p(\varphi) + W^p(\xi)$ turn out to be solutions of the homogeneous problem (4.39). The Green formula leads to

$$\begin{aligned} &\int_{\partial\omega} W^q \partial_\nu W^p ds_\xi - \int_{\partial\omega} \rho^{q/2} \Phi_q \partial_\nu (\rho^{p/2}\Phi_p) ds_\xi \\ &= \int_{\partial\omega} (Y^q \partial_\nu W^p - W^p \partial_\nu Y^q) ds_\xi = \int_{\{\xi \in \mathbb{K}: \rho = R\}} (W^p \partial_\rho Y^q - Y^q \partial_\rho W^p) ds_\xi \\ &= \sum_{j=1}^{3} R^{\pi(q-j)} \frac{m_{pj}}{2\pi j} (q+j) \int_{-\pi}^{\pi} \Phi_q(\varphi) \Phi_j(\varphi) d\varphi + o(1). \end{aligned} \tag{4.42}$$

Since the left-hand side equals the sum of scalar products in (4.41), the right-hand side has the finite limit, which in view of the definition of the angular parts Φ_k equals to $\sum \mathbf{m}_{pj}\delta_{j,q} = \mathbf{m}_{pq}$. Therefore, the matrix $\mathbf{m}$ takes the form of the sum of two Gram matrices, symmetric and positive definite. ■

Remark 4.5. In the case of the kinked crack $\mathrm{mes}_2\omega = 0$ and therefore, the second term on the right-hand side of (4.41) vanishes. Nevertheless, the matrix $\mathbf{m}$ of Lemma 4.2 keeps the properties. We refer to [34, 1], for much more involved theory of cracks elongation in elastic solids. ■

In view of formulae (4.37), (4.38) and (4.40), in the same way as it is described in Section 2.3, we can formulate boundary value problems for terms $v_m^{(2)}$ and $v_m^{(3)}$ of regular type

$$\begin{aligned} -\Delta_x v_m^{(q)}(x) &= \lambda_m^0 v_m^{(q)}(x) + \lambda_m^{(q)} v_m^0(x) + [\Delta_x, \chi(x)] T_m^q(x) \\ &\quad + \lambda_m^0 \chi(x) T_m^q(x), \quad x \in \Omega, \quad \partial_m v_m^{(q)}(x) = 0, \quad x \in \partial\Omega; \end{aligned} \tag{4.43}$$

where

$$\begin{aligned} T_m^2(x) &= \pi^{-1} K_m^1 m_{11} r^{-1/2} \Phi_1(\varphi), \\ T_m^3(x) &= \pi^{-1} K_m^2 m_{21} r^{-1/2} \Phi_1(\varphi) + (2\pi^{-1}) K_m^1 m_{12} r^{-1/2} \Phi_2(\varphi). \end{aligned} \tag{4.44}$$

The compatibility conditions for problems (4.43) with $q = 2, 3$, are processed by the method [29]: the Green formula is applied in the domain $\Omega \setminus \mathbb{B}_\rho$ and taking into account the asymptotic decomposition (4.36) the integral on the contour $(-\pi, \pi) \ni \varphi$ is evaluated (compare with the procedure in (4.42)). As a result, by the normalisation (1.9), the following expressions are obtained

$$\lambda_m^{(2)} = \mathbf{m}_{11}(K_m^1)^2, \quad \lambda_m^{(3)} = 2K_m^1\mathbf{m}_{12}K_m^2. \tag{4.45}$$

Now we use the theory of elliptic boundary value problems in the domains with corner points, taking into account that the right-hand sides (4.44) of problem (4.43) for $q = 2$ and $q = 3$ are only of order $O(r^{-1/2})$ and $O(r^{-1})$, respectively, therefore

$$\begin{aligned} v_m^{(2)}(x) &= K_m^{12}r^{1/2}\Phi_1(\varphi) + K_m^{22}r^1\Phi_2(\varphi) + O(r^{3/2}), \\ v_m^{(3)}(x) &= K_m^{13}r^{1/2}\Phi_1(\varphi) + O(r(1 + |\ln r|)) \end{aligned} \tag{4.46}$$

We refer the reader, e.g., to [36, Ch. 2] for all details which are needed to derive (4.46); see also Remark 2.2. The next step of our procedure is the formulation of problems for $w_m^{(3)}$ and $w_m^{(4)}$:

$$-\Delta_\xi w_m^{(q)} = 0, \quad \xi \in \Xi, \quad \partial_\nu w_m^{(q)}(\xi) = G_m^{(q)}(\xi), \quad \xi \in \partial\Xi. \tag{4.47}$$

We use the following notation

$$G_m^{(3)}(\xi) = -K_m^3\partial_n(\rho^{3/2}\Phi_3(\varphi)) - K_m^{12}\partial_\nu(\rho^{1/2}\Phi_1(\rho)),$$

$$G_m^{(4)}(\xi) = -K_m^4\partial_n(\rho^2\Phi_4(\varphi)) - K_m^{22}\partial_\nu(\rho^1\Phi_2(\rho)) - K_m^{13}\partial_n(\rho^{1/2}\Phi_1(\varphi)) + \frac{1}{4}\lambda_m^0 v^0(O)\partial_\nu\rho^2.$$

In this way, for $q = 3$ we have $w_m^{(3)}(\xi) = K_m^3 W^3 + K_m^{12}W^1$, moreover

$$w_m^{(3)}(\xi) = (K_m^3 m_{31} + K_m^{12}m_{11})\rho^{-1/2}\Phi_1(\rho) + O(\rho^{-1}), \quad \rho \to +\infty, \tag{4.48}$$

and, thus, the detached asymptotics term has to be put into the problem for $v_m^{(4)}$. We observe that

$$\int_{\partial\omega\cap\mathbb{K}} G_m^{(4)}(\xi)ds_\xi = \frac{1}{4}\lambda_m^0 v^0(O) \int_{\partial\omega\cap\mathbb{K}} \partial_\nu\rho^2 ds_\xi = -\lambda_m^0 v^0(O) mes_2\omega.$$

Thus, problem (4.47) for $q = 4$ has no decaying solution, there exists a solution with the logarithmic growth

$$w_m^{(4)}(\xi) = -\lambda^0 v^0(O) mes_2\omega \frac{1}{2\pi}\ln\rho + O(\rho^{-1}), \quad \rho \to \infty. \tag{4.49}$$

In the next step of the procedure we proceed in very similar way described in §2, but with the ansätze (4.37), (4.38) and with decompositions (4.40), (4.48), (4.49), as a result the following boundary value problem is obtained for $v_m^{(4)}$

$$\begin{aligned} -\Delta_x v_m^{(4)}(x) &= \lambda_m^0 v_m^{(4)}(x) + \lambda_m^{(2)} v_m^{(2)}(x) + \lambda_m^{(4)} v_m^0(x) + [\Delta_x, \chi(x)]T_m^4(x) \\ &\quad + \lambda_m^0\chi(x)T_m^4(x), \quad x \in \Omega, \quad \partial_n v_m^{(4)}(x) = 0, \quad x \in \partial\Omega. \end{aligned} \tag{4.50}$$

Here the notation is used

$$\begin{aligned} T_m^4(x) &= K_m^1\frac{m_{13}}{3\pi}r^{-3/2}\Phi_3(\varphi) + K_m^2\frac{m_{22}}{2\pi}r^{-1}\Phi_2(\varphi) \\ &\quad + (K_m^3\frac{m_{31}}{\pi} + K_m^{12}m_{11})r^{-1/2}\Phi_3(\varphi) - \lambda_m^0 v_m^0(O) mes_2\omega\frac{1}{2\pi}\ln\frac{r}{\varepsilon}. \end{aligned} \tag{4.51}$$

The solutions $v_m^{(2)}$ and $v_m^{(3)}$ of problem (4.43) is defined up to the term $cv_m^{(0)}$, i.e. we can require that the following condition is satisfied

$$(v_m^{(q)}, v_m^0)_\Omega = 0, \quad q \geq 1, \tag{4.52}$$

which provides the uniqueness of the intensity factors K_m^{12} and K_m^{13} in decompositions (4.46). In view of formulae (4.44) for T_m^2 and (4.45) for $\lambda_m^{(2)}$ the function $v_m^{(2)}$ is proportional to the intensity coefficient K_m^1, which means that

$$K_m^{12} = \mathcal{K}_m K_m^1, \tag{4.53}$$

where $\mathcal{K}_m$ is a constant, which depends on the domain Ω and on the selected simple eigenvalue λ_m^0.

Taking into account formulae (4.51)-(4.53), and according to the method of [29] the compatibility conditions for problem (4.54) can be formulated as follows

$$\lambda_m^{(4)} = 2m_{13}K_m^1K_m^3 + m_{22}(K_m^2)^2 + m_{11}\mathcal{K}_m(K_m^1)^2 - \lambda_m^0 v_m^0(O) mes_2\omega. \tag{4.54}$$

We do not justify the asymptotics, since the related theorem can be established in the same way as it is described in details in §3. We observe only, that decomposition (4.37) with the four terms in precise of the order at least $O(h^{5/2}(1+|\ln h|))^{5/2}$ (compare with (3.34)).

We rewrite formulae (4.37) and (4.45), (4.54) in the following form

$$\begin{aligned}\lambda_m^h = \lambda_m^0 &+ \sum_{k=1}^{2} m_{jk} h^{(j+k)/2} K_m^j K_m^k \\ &+ h^2\left\{2m_{13}K_m^1K_m^3 + m_{11}\mathcal{K}_m(K_m^1)^2 - \lambda_m^0 v_m^0(O)^2 mes_2\omega\right\} + O(h^{5/2}(1+|\ln h|))^{5/2}.\end{aligned} \tag{4.55}$$

Remark 4.6. It is possible to add to the right-hand side of (4.55) the higher order terms $2h^{5/2}m_{23}K_m^2K_m^3$ and $h^3m_{33}(K_m^3)^2$. The formula with exactly same asymptotic precision reads

$$\begin{aligned}\lambda_m^h = \lambda_m^0 &+ \sum_{j,k=1}^{3} m_{jk} h^{(j+k)/2} K_m^j K_m^k \\ &+ h^2 m_{11}\mathcal{K}_m(K_m^1)^2 - h^2\lambda_m^0 v_m^0(O)^2 mes_2\omega + O(h^{5/2}(1+|\ln h|))^{5/2}.\end{aligned} \tag{4.56}$$

As it follows from the proof of Lemma 4.2, 3×3-matrix $(\mathbf{m}_{jk})_{j,k=1}^3$ is again symmetric and positive definite. If the asymptotic terms of lower order are taken into account, it necessitates an extension of the matrix of decomposition coefficients (4.40), the appropriate construction is described in [33], [16] for the domain perturbation by small opening, and the passage to the crack with $\alpha = 2\pi$ can be performed by an application of the method proposed in [37]. ■

4.4. **Growing of geometrical domain.** In the analysed already perturbations, it was always assumed that the perturbations of the domain result in a decrease of the volume, i.e., $mes_2\Omega(h) \le mes_2\Omega$. However in the shape optimisation it is always possible to require also that the volume of domain increases. If Ω is a domain with smooth boundary, then $mes_2\Omega(h) > mes_2\Omega$ in the case of

$$\Xi = \mathbb{R}^2_- \cup \omega, \tag{4.57}$$

$$\Omega(h) = (\Omega \setminus \mathcal{U}) \cup \left\{x \in \mathcal{U} : \xi = (h^{-1}n, h^{-1}s) \in \Xi\right\} \tag{4.58}$$

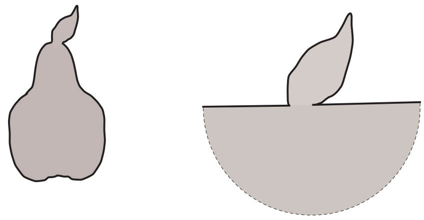

FIGURE 7. The growing of the perturbed domain.

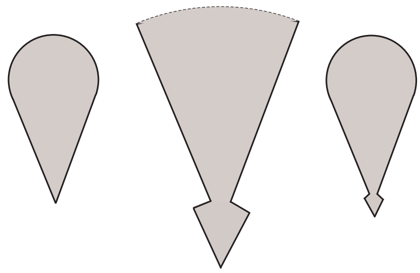

FIGURE 8. Exterior perturbation of the corner point.

4.5. **Spectral problem for a crack.** (see Figures 7 and 8). Here ω is a domain such that $\omega \cap \mathbb{R}^2_+ \neq \emptyset$, and $\mathcal{U}$ is a neighbourhood of the origin. We assume that the set (4.58) is connected and its boundary $\partial\Omega(h)$ is piecewise smooth.

Asymptotic procedures presented in §2 and in Section 4.1, can be applied without any substantial modification, and the justification of asymptotics given in §3 simplifies in the present case due to the fact, that we do not need any extension of eigenfunctions from the domain (1.2) onto Ω, now we simply use the restriction of the eigenfunction defined on the set (4.58) to Ω. We also point out that in the asymptotic ansatz (1.11) we need a smooth extension of the function v^0 onto an open neighbourhood of the set $\overline{\Omega}$, so now we can restrict to Ω the function defined on (4.58). In addition, for the extended function $\widehat{v}^0$ the representations (2.1) and (4.6) are still valid, therefore the problems for the boundary layer type terms of our ansätze are of the same form as before.

We recall that for the solution v^2 of Neumann problem (2.39), (2.40) the extension in the class $H^1(\Omega)$ is needed, with the property that the extended function is bounded. In the case of the Dirichlet problem (2.39), (4.15) in general no extension of the solution v^2 is required, since the solution is multiplied by a cut-off function X_h in the global asymptotic approximation for the spectral problems (1.3), (4.1) or (1.3), (4.2) (see the subtrahend inside of the norm on the right-hand side of (4.23)), and the function X_h equals to zero on the set $\Omega(h) \setminus \Omega$. If Ω is the domain with a corner point on the boundary, and $\Omega(h) = \Omega \cup \omega_h$ where ω_h is a small set (4.24) (see Figure 8 and compare with section 4.2). Then the extension of function (4.25) is not always available, so it is necessary to multiply the members of the asymptotic ansatz by a cut-off function X_h. Such an approach is common in the theory of elliptic boundary value problems in domains with singularly perturbed boundaries [28] and it is described in details in Chapter 4 of this monograph, (see also the original paper [25]).

4.6. **Mixed boundary value problems with the Dirichlet boundary conditions on the boundary of a cavern.** The most complicated asymptotic ansätze appear for the boundary conditions

$$\partial_n u^h(x) = 0, \quad x \in \partial\Omega(h) \cap \partial\Omega, \quad u^h(x) = 0, \quad x \in \partial\Omega(h) \setminus \partial\Omega. \tag{4.59}$$

In such a case the principal feature which causes serious difficulties is but the lack of the decay at infinity of solutions to the limit problem

$$\begin{aligned} -\Delta_x w^0(\xi) = 0, \quad \xi \in \Xi, \quad \partial_{\xi_1} w^0(\xi) = 0, \quad \xi \in \partial\Xi \cap \partial\mathbb{R}^2_-, \\ w^0(\xi) = g^0(\xi), \quad \xi \in \partial\Xi \setminus \mathbb{R}^2_-, \end{aligned} \tag{4.60}$$

even for the function g^0 with the null mean value, therefore the decay of boundary layer terms is produced artificially. Complications appear already at the stage of construction of principal members of ansätze, thus the question on the dependence of asymptotic structures on the curvature pass to the second plan. Moreover, the algorithm of construction of asymptotics and its justification differ only in some details with the framework given in [27] (see also [28, Ch. ?]) for the Dirichlet problem in the domain with small interior opening. We recall here, that the complexity of asymptotic constructions for solutions to problem (1.3), (4.59) could causes even some mistakes in published results (for exemple see [31] and the explanation given in [5]). For the convenience of the reader we briefly explain the algoritm of [27] in our context.

Assume that $\lambda^0 = \lambda^0_m$ is a simple eigenvalue of problem (1.8), and let $v^0 = v^0_m$ be the associated eigenfunction normalized in $L_2(\Omega)$, and such that $v^0(O) \neq 0$. The case of $m = 0$ is not excluded, i.e., $\lambda^0 = 0$ and $v^0(x) = (mes_2\Omega)^{-1/2}$.

We need in the sequel certain special solutions of limit boundary value problems which we list now. The first special solution is the generalized Green function [43] with the singularity at the point $O \in \partial\Omega$, namely the solution to the problem

$$-\Delta_x \mathcal{G}_m(x) = \lambda^0_m \mathcal{G}_m(x) - v^0_m(O) v^0_m(x), \quad x \in \Omega, \quad \partial_n \mathcal{G}_m(x) = \delta(x), \quad x \in \Gamma. \tag{4.61}$$

The Green function is smooth in $\overline{\Omega} \setminus O$, verifies the orthogonality condition $(\mathcal{G}_m, v^0_m)_\Omega = 0$, and in the vicinity of the singular point O it admits the decomposition

$$\mathcal{G}_m(x) = -\pi^{-1} \ln r + \mathcal{G}^0_m + O(r), \quad r \to 0. \tag{4.62}$$

The second special solution is the so-called capacitary potential $\mathcal{E}$ (see [22]) of the set (2.25), i.e. $\mathcal{E}$ is a harmonic function, $\mathcal{E} = 0$ on the curve $\partial\Xi \setminus \partial\mathbb{R}^2_-$, its normal derivative $\partial\mathcal{E}/\partial\xi_1 = 0$ on the set $\partial\Xi \cap \partial\mathbb{R}^2_-$, and $\mathcal{E}$ admits the decomposition

$$\mathcal{E}(\xi) = -(2\pi)^{-1} \ln\rho + \mathcal{E}_0 + \widehat{\mathcal{E}}(\xi), \quad \widehat{\mathcal{E}}(\xi) = O(\rho^{-1}), \quad \rho \to \infty. \tag{4.63}$$

In the literature [22], [41], the quantity $\exp(2\pi\mathcal{E}_0)$ is called the logarithmic capacity or exterior conformal radius of the set Ξ^{00}. In order to avoid the presence of cut-off function X_h in all asymptotic formulae, we simply assume that $O \notin \overline{\Omega(h)}$.

Now, we turn to the asymptotic ansatz, which according to [27], can be written in the following way

$$\lambda^h_m = \lambda^0_m + \Lambda^0_m(z_m) + \dots, \tag{4.64}$$

$$u^h_m(x) = v^0_m(x) + a_m(z_m)\mathcal{G}_m(x) + V^0_m(x, z_m) + \chi(x) w^0_m(\xi, z_m) + \dots \tag{4.65}$$

Here, $z_m = (|\ln h| + z^0_m)^{-1}$ is a new parameter, and the quantities z^0_m, $\Lambda^0_m(z)$, and $a_m(z)$, are to be determined. The functions w^0_m and V^0_m decay for $\rho \to \infty$ and $r \to 0$, respectively, in particular we have

$$V_m^0(O, z_m) = 0. \tag{4.66}$$

In view of formulae (4.62) and (4.66), the right-hand side of boundary conditions in problem (4.60), collecting the principal terms of discrepancies resulting in the second condition of (4.59) from the first three members in ansatz (4.65), takes the form

$$g^0(\xi, z_m) = -v_m^0(O) - a_m(z_m)(\pi^{-1}\ln h - \pi^{-1}\ln\rho + \mathcal{G}_m^0).$$

According to (4.63) the solution of problem (4.60) with a such right-hand side is given by

$$w^0(\xi, z_m) = 2a_m(z_m)\widetilde{\mathcal{E}}(\xi) + a_m(z_m)(\pi^{-1}|\ln h| + 2\mathcal{E}_0 - \mathcal{G}_m^0) - v_m^0(O).$$

The solution decays at infinity provided that

$$a_m(z_m) = \pi v_m^0(O)\Big[|\ln h| + 2\pi\mathcal{E}_0 - \pi\mathcal{G}_m^0\Big]^{-1} =: \pi v_m^0(O) z_m. \tag{4.67}$$

Let us consider now the problem for $V_m^0(x, z_m)$, which can be determined according to ansätze (4.64), (4.65), with the decaying term w_m^0, and in view of equalities (4.61) for $\mathcal{G}_m$, it takes the form

$$\begin{aligned} -\Delta_x V_m^0(x, z_m) - \lambda^0 V_m^0(x, z_m) &= F_m^0(x, z_m) \\ := \Lambda_m^0(z_m)V_m^0(x, z_m) + \Lambda_m^0(z_m)&v^0(x) + \Lambda_m^0(z_m)a_m(z_m)\mathcal{G}_m(x) \\ -a_m(z_m)v_m^0(O)v_m^0(x), &\quad x \in \Omega, \end{aligned} \tag{4.68}$$

$$\partial_n V_m^0(x, z_m) = 0, \quad x \in \partial\Omega, \tag{4.69}$$

Lemma 4.3. *Assume that the eigenfunction v_m^0 is associated to a simple eigenvalue, it is normalised in $L_2(\Omega)$, and in addition $v_m^0(O) \neq 0$. For all right-hand sides $F \in L_2(\Omega)$, with the orthogonality condition $(F, v_m^0)_\Omega = 0$, the Neumann problem*

$$\Delta_x V - \lambda_m^0 V = F \quad \text{in} \quad \Omega, \quad \partial_n V = 0 \quad \text{on} \quad \partial\Omega$$

admits the unique solution $V \in H^2(\Omega)$, with $V(O) = 0$. Furthermore, $\|V; H^2(\Omega)\| \leq c\|F; L_2(\Omega)\|$.

Proof. The existence of a particular solution $V^\bullet$ and its differential properties are well known (see e.g., [21]). The assumption $v_m^0(O) \neq 0$ allows us to determine the constant c_0 in the formula for the general solution $V = V^\bullet + c_0 v_m^0$ by the additional condition $V(O) = 0$. ⊠.

Hence, assuming that the right-hand side F_m^0 in (4.68) is given, we can conclude that the solution V_m^0 verifies condition (4.68) if and only if

$$\Lambda_m^0(z_m)(V_m^0, v_m^0)_\Omega + \Lambda_m^0(z_m) - a_m(z_m)v_m^0(O) = 0$$

or, which is equivalent, according to (4.67)

$$\Lambda_m^0(z_m) = \pi|v_m^0(O)|^2 z_m(1 + (v_m^0, v_m^0)_\Omega)^{-1}. \tag{4.70}$$

We inject (4.70) into (4.68), hence V_m^0 is a solution of nonlinear problem, with the Neumann condition (4.69) and the Sobolev condition (4.66), defined by a mapping from the subspace

$$\mathcal{H} = \Big\{V \in H^2(\Omega) : \partial_n V = 0 \quad \text{on} \quad \partial\Omega, \quad V(O) = 0\Big\}$$

into the subspace

$$\mathcal{L} = \Big\{F \in L_2(\Omega) : (F, v_m^0)_\Omega = 0\Big\}.$$

Since the nonlinear perturbation of the isomorphism $\mathcal{H} \approx \mathcal{L}$ defined in Lemma 4.3 turns out to be a small and analytic perturbation, general results [20] show that for sufficiently small values of parameter z_m introduced in (4.67), problem (4.68)-(4.70), (4.66) has the unique solution V_m^0, analytical with respect to z_m, and such that $V_m^0(x;0) = 0$. We inject the solution into (4.70) and obtain the correction $\Lambda_m^0(z_m)$ in the asymptotic ansatz (4.64), the resulting correction is analytical with respect to z_m, moreover

$$\Lambda_m^0(z_m) = \pi v^0(O)^2 z_m + O(z_m^2). \tag{4.71}$$

In this way the formal asymptotic analysis is performed, and we refer the reader to [27] for the arguments on the justification of asymtotics.

Remark 4.7. It is not difficult to construct the expansion of the eigenvalue λ_m^n in the series of inverse powers of the large parameter $|\ln h|$

$$\lambda_m^n \sim \sum_{p=0}^{\infty} |\ln h|^{-p} \Lambda_m^{(p)} . \tag{4.72}$$

The results already established show that the series converges, and the remainder is of the order $O(h^{1-\delta}), \delta > 0$. On the other hand, the function $h \to |\ln h|$ is slowly increasing, hence the expansion (4.72), and in particular the resulting form (4.64), (4.71) formula

$$\lambda_m^h = \lambda_m^0 + |\ln h|^{-1}\pi |v_m^0(O)|^2 + O(|\ln h|^{-2}) \tag{4.73}$$

is not sufficiently precise, and therefore its utility in shape optimisation is questionable. ■

Remark 4.8. If λ_m^0 is a simple eigenvalue but the corresponding eigenfunction takes the value $v^0(O) = 0$, the asymptotic analysis is performed in exactly the same way as it is described in §2, i.e., it looses the complication discussed above. For a multiple eigenvalue (see (2.48)) the eigenfunctions $v_m^0, \dots, v_{m+\kappa_m-1}^0$ can be fixed in such a way that the relations (see (1.9)) are verified and $v_{m+1}^0(O) = \cdots = v_{m+\varkappa_m-1}^0(O) = 0$. In this way, at most one between eigenvalues $\lambda_m^h, \dots, \lambda_{m+\varkappa_m-1}^h$ in problem (1.3), (4.59) requires the complicated asymptotic analysis. ■

5. On shape optimisation

5.1. **Reduction and enhancement of eigenvalues (eigenfrequences) by boundary perturbations.** For all three problems (1.3), (1.4) or (1.3), (4.1) and (1.3), (4.2), formulae (2.47) and (4.20) for the asymptotic correction of a simple eigenvalue λ_m^0 can be presented in a unique way with the help of the corresponding eigenfunction v_m^0 normalized in $L_2(\Omega)$:

$$\lambda_m^h = \lambda_m^0 + h^2 \left(\mathbf{m}(\Xi) |\nabla_x v^0(O)|^2 - \lambda_m^0 mes_2\omega |v^0(O)|^2 \right) + O(h^{3-\delta}). \tag{4.74}$$

Here, δ is an arbitrary positive number for (1.4) and $\delta = 0$ for cases (4.1) and (4.2). Thus, the sign of the multiplier for h^2 in (1.9) is determined according the way the boundary is perturbed, energy $\mathbf{m}(\Xi)$ and geometry $mes_2\omega$ characteristics, as well as by the location of the point O on the contour Γ.

5.2. **Control on eigenvalues: Dirichlet problem.** If, as a result of the boundary perturbations a cavern is formated, from Lemma 6.1 it follows that the multiplier $\mathbf{m}$ in (4.74) is negative. In view of the equality $v^0(0) = 0$ this implies $\lambda_m^h < \lambda_m^0$ under the condition $\partial_n v_m^0(O) \neq 0$. We point out that the same conclusion can be drown out from the minimax principle (cf. [47])

$$\lambda_m^h = \min_{\mathcal{R}_m \subset \mathcal{H}_h} \max_{v \in \mathcal{R}_m} \frac{\|\nabla_x v; L_2(\Omega(h))\|^2}{\|v; L_2(\Omega(h))\|^2} . \tag{4.75}$$

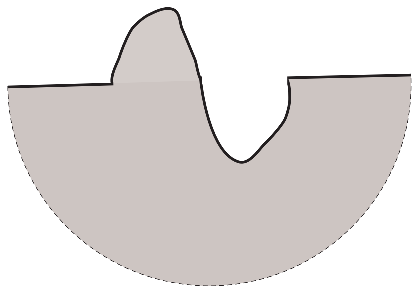

FIGURE 9. The perturbation of the domain with the area preserved.

Here

$$\mathcal{H}_h = \mathring{H}^1(\Omega(h)) = \left\{ u \in H^1(\Omega(h)) : u = 0 \quad \text{on} \quad \partial\Omega(h) \right\}$$

with $\mathcal{H}_0 = \mathring{H}^1(\Omega)$, and by $\mathcal{R}_m$ is denoted a subspace of dimension m. We observe that the required result is obtained by the embedding $\mathring{H}^1(\Omega(h)) \subset \mathring{H}^1(\Omega)$, with the convention that the functions are extended by zero on $\Omega \setminus \Omega(h)$, therefore $\lambda_m^h > \lambda_m^0$.

Assume that $\Omega(h)$ is the domain from (4.58), and Ξ is the domain which coincides with the half-plane R^2 outside of the circle $\mathbb{B}_{R^0}$. We denote $\omega_+ = \Xi \setminus \overline{\mathbb{R}^2_-}$ and $\omega_- = \Xi \cap \mathbb{R}^2_-$ (see Fig. 9). In this case principle (4.75) becomes useless. From the explications given in section 6.3 it follows that formulae (4.20) and (4.22) are valid also in the case when the both sets ω_+ and ω_- are not empty. In addition, a natural modification of the proof of Lemma 4.1 leads to the following variant of formula (4.12):

$$\mathbf{m}(\Xi) = -\int_{\Xi} |\nabla_\xi W(\xi)|^2 d\xi - mes_2\omega_- + mes_2\omega_+ \,. \tag{4.76}$$

Hence, $m(\Xi) < 0$ under the condition that $mes_2\omega_- \geq mes_2\omega_+$ (the volume of added part is greater then the volume of the clipped part). If $mes_2\omega_+ > mas_2\omega_-$, it is possible to predict the sign of quantity (4.76) only under additional assumption that $\omega_- = \emptyset$.

Lemma 5.1. *If the domain Ξ is of form (4.57) and the set $\omega_+ = \omega \cap \mathbb{R}^2_-$ is nonempty, then the multiplier $\mathbf{m} = \mathbf{m}(\Xi)$ in the asymptotics (4.11) of the solution to problem*

$$-\Delta_\xi W(\xi) = 0, \quad \xi \in \Xi, \quad W(\xi) = -\xi_1, \quad \xi \in \partial\Xi, \tag{4.77}$$

is positive.

Proof. The right-hand side of boundary condition in (4.77) is negative on $\partial\omega \setminus \overline{\mathbb{R}^2_-}$ and is null on $\partial\mathbb{R}^2_- \setminus \omega$. The maximum principle assures that $W(\xi) < 0$ in Ξ, which means $\mathbf{m} \geq 0$ since the harmonic function $\rho^{-2}\xi_1$ is negative in half-space $\mathbb{R}^2_-$. The equality $\mathbf{m} = 0$ is impossible, since all harmonics decaying for $\rho \to \infty$ change the sign in infinity. ■

We note that, under growing of the domain , the multiplier $\mathbf{m}(\Xi)$ in expansions (4.11) and (2.21) and in others, cannot be express through standard integral attributes of sets [41].

5.3. **Control on eigenvalues: Neumann problem.** According to formula (2.24) for problem (1.3), (1.4) in domain (1.2) with a cavern the coefficient $\mathbf{m}(\Xi)$ is positive, however the differences $\lambda_m^h - \lambda_m^0$ for $m \geq 1$ can have arbitrary sign in dependence of the position of the point O. Indeed, let O coincide, for example, with a local extremum of the function

$$\Gamma \ni s \longmapsto v_m^0(0, s). \tag{4.78}$$

Then $|\nabla_x v_m^0(O)| = 0$ in view of homogeneous Neumann condition for v_m^0 thus $\lambda_m^h - \lambda_m^0 < 0$, provided that $v_m^0(O) \neq 0$ and the parameter h is sufficiently small (if O is the global minimum, then the inequality $v_m^0(O) \neq 0$ can be established by the maximum principle). Taking off a cavern at the point O of sign change of function (4.78), and assuming that $\partial_s v_m^0(O) \neq 0$ (for the eigenfunction associated to the first positive eigenvalue λ_1^0 such an assumption is satisfied) we obtain that $\lambda_m^h - \lambda_m^0 > 0$ for h sufficiently small. If the perturbed domain $\Omega(h)$ contains Ω and takes form (4.58) (see Fig. 9), unfortunately the sign of $\mathbf{m}(\Xi)$ is not known, the coefficient $\mathbf{m}(\Xi)$ in view of (2.22) and (2.23) is obtained in the analogous to (2.24) and (4.76) form

$$\mathbf{m}(\Xi) = \int_{\Xi} |\nabla_\xi W(\xi)|^2 d\xi + mes_2\omega_- - mes_2\omega_+ \ . \tag{4.79}$$

Indeed, for $\omega_- = \varnothing$ and $mes_2\omega_+ > 0$ in the right-hand side of (4.79) appears a difference of unknown sign. We emphasise that the proof of Lemma 5.1 based on the maximum principle cannot be used, in addition the conformal mapping method indicated in Remark 4.4 is also not applicable.

5.4. **Mixed boundary value problems.** For mixed boundary value problems under boundary conditions (4.2) the minimax principle (4.75) is not applicable; however in view of Lemma 4.1 the constant $\mathbf{m}(\Xi)$ is positive for a cavern, hence the same conclusion as for the Neumann (1.3), (1.4) problem is valid in view of the equality $v_m^0(O) = 0$. If $\Omega(h) \not\subset \Omega$ formula (4.79) holds, which means that for $mes_2\omega_- \geq mes_2\omega_+$ it follows that $m(\Xi) > 0$ as before. On the other hand, for $\omega_- = \varnothing$ the sign of the quantity $m(\Xi)$ is unknown.

For boundary condition (4.59), the priciple (4.75) leads to the inequality $\lambda_m^h > \lambda_m^0$ provided that $\Omega(h) \subset \Omega$. If the domain $\Omega(h)$ is of form (4.58), the strict inequality holds at least for simple eigenvalues by formula (4.73), however only for sufficiently small h since it is required that the term $|\ln h|^{-1}$ is small enough.

5.5. **Multiple eigenvalues.** Assume that λ_m^0 is a multiple eigenvalue (see (2.48)) of the Dirichlet problem (4.4). Then the asymptotic corrections $\lambda^{m\prime}, \dots, \lambda^{m+\varkappa_m-1\prime}$ in ansatz (1.10) for problem (1.3), (4.1) or (1.3), (4.2) are given by eigenvalues $(\varkappa_m \times \varkappa_m)$-matrix $\mathbf{M}$ with elements (4.21). Since $\mathbf{M}$ is proportional to a matrix of the form $\mathcal{M}\mathcal{M}^T$, where $\mathcal{M}$ is a column of heigth $\varkappa_m$, and $\mathcal{M}^T$ is the transposed row, hence the eigenvalues take the form

$$\lambda^{m\prime} = \mathbf{m}(\Xi) \sum_{p=m}^{m+\varkappa_m-1} |\partial_n v_p^0(O)|^2, \quad \lambda^{m+1\prime} = \dots \lambda^{m+\varkappa_m-1\prime} = 0. \tag{4.80}$$

Therefore, Theorem 6.3 assures a nontrivial correction for the only one among $\varkappa_m$ eigenvalues. There are known standard procedures (see, e.g., [28]), which allow to construct the higher order terms of expressions (1.10) and (1.11) and to determine if the eigenvalues decrease of increase in terms of higher order corrections.

For the Neumann problem (1.3), (1.4) the matrix $\mathbf{M}$ with elements (2.53) turns out to be the sum of two matrices of the form $\mathcal{M}\mathcal{M}^T$ thus the eigenvalues are not given in the simple form (4.80).

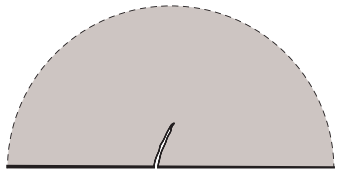

FIGURE 10. The selvage micro-crack.

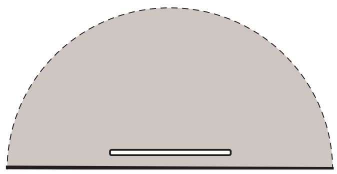

FIGURE 11. The micro-crack parallel to the boundary.

5.6. **Corner point.** All theory explained up to now is valid for problems from Section 4 (see Figures 5 and 6), when the limit domain has a corner point at the boundary. We note only that the sum with respect to $j, k = 1, 2, 3$ in (4.56) with the definite matrix $\mathbf{m}_{jk}$, shows the more general property of monotone dependence of eigenvalues with respect to boundary perturbations

5.7. **Selvage micro-crack.** Let $\Upsilon := \overline{\omega}$ is a segment of a curve, i.e., $\Xi = \mathbb{R}^2_- \setminus \Upsilon$ is a half-plane with a crack (see Figures 10 and 11).

In the case $mes_2\omega = 0$ and formulae (1.10), (2.47) and (3.34) for eigenvalues (simple eigenvalues are considered here, the case of multiple eigenvalues is discussed in section 5.5) of the problem (1.3), (1.4) can be reduced to

$$\lambda^h_m = \lambda^0_m + h^2 |\partial_n v^0_m(O)|^2 \int\limits_{\Xi} |\nabla_\xi W(\xi)|^2 d\xi + O(h^3 (1 + |\ln h|)^{\frac{5}{2}}). \tag{4.81}$$

Here W is the solution (2.21) to problem (2.20), which equals to zero only in the case of a straight crack Υ parallel to the boundary $\partial\mathbb{R}^2_-$ (see Fig. 11), since in such a case $\nu_2 = 0$

on the surfaces $\Upsilon^{\pm}$ of the cut. If Υ is a boundary of curvilinear crack (Fig. 10), then the Dirichlet integral in (4.81) is positive. The fact is in accordance with the minimax principle (4.75) for sequences (1.5) and (1.7): where $\mathcal{R}_m$ is $(m+1)$-dimensional subspace of the Sobolev spaces $H^1(\Omega(h))$ and $H^1(\Omega) \subset H^1(\Omega(h))$.

Formulae (4.81) can be employed for solving one more shape optimisation problem. Let λ_1^0 and λ_2^0 be simple eigenvalues and it is required to find out the location of a crack of the length h, such that the distance between λ_1^h and λ_2^h is maximised. It follows that

$$\lambda_2^h - \lambda_1^h = \lambda_2^0 - \lambda_1^0 + h^2 \int_{\Xi} |\nabla_\xi W(\xi)|^2 d\xi (|v_2^0(O)|^2 - |v_1^0(O)|^2) + O(h^3(1+|\ln h|))^{\frac{5}{2}}.$$

In such way, for a small h, the crack should be located at the maximum of the function

$$\Gamma \ni s \longrightarrow v_2^0(s)^2 - v_1^0(s)^2.$$

Let us observe, that the integral $\int |\nabla_\xi W|^2 d\xi$ attains its maximal value for the crack of unit length orthogonal to the boundary (see [42]). Similar and related problems can be analysed and solved for other types of boundary perturbations and for the other boundary conditions.

5.8. **Asymptotics of energy functionals.** In domain (1.2) the Dirichlet problem is considered

$$-\Delta_x u^h(x) = f(x), \quad x \in \Omega(h), \quad u^h(x) = 0, \quad x \in \Gamma(h) = \partial\Omega(h), \tag{4.82}$$

and the functional evaluated for its solution

$$\mathcal{T}(u^h; \Omega(h)) = \int_{\Omega(h)} T(u^h(x), x) dx. \tag{4.83}$$

For the sake of simplicity here we assume that all data are given by smooth functions, i.e., $f \in C^\infty(\overline{\Omega})$, $T \in C^\infty(\mathbb{R} \times \overline{\Omega})$, and the boundaries Γ and $\partial\Xi$ of the limit domains are also smooth. Of course, these assumptions can be easily omitted by the technique of the previous sections. It is obvious that we can assume that $T(0; x) = 0$.

The goal of the section is to establish the asymptotics of functional (4.83) for $h \to +0$. The construction and justification of asymptotics of the solutions to problem (4.82) do not require any new argument compared to employed already in previous sections: asymptotic approximation of the solution u^h is given by

$$U^h(x) = v^0(x) + h\chi(x)w^1(\xi) + h^2\chi(x)w^2(\xi) + h^2\chi_h(x)v^2(x), \tag{4.84}$$

where χ and χ_h are cut-off functions, present in formulae (1.11) and (4.23), and v^0 stands for solution of the limit problem

$$-\Delta_x v^0(x) = f(x), \quad x \in \Omega, \quad v^0(x) = 0, \quad x \in \Gamma = \partial\Omega. \tag{4.85}$$

Decomposition (4.7) is valid, hence the boundary layer terms w^1 and w^2 are given by solutions to (4.7), (4.8) and (4.16)-(4.18), respectively. The problems for boundary layers admit solutions decaying at infinity with the rate $O(\rho^{-1})$, in addition, for w^1 the representation (4.10), (4.11) are valid, on contrary, the representation for w^2 is not applied in the sequel. Finally, v^2 stands for the solution of the problem

$$-\Delta_x v^2(x) = f^2(x), \quad x \in \Omega, \quad v^2(x) = 0, \quad x \in \Gamma, \tag{4.86}$$

with the right-hand side

$$f^2(x) = \Delta_x(\chi(x)t^1(n,s)), \quad t^1(n,s) = \frac{m(\Xi)}{\pi}\partial_n v^0(O)\frac{n}{n^2+s^2}. \tag{4.87}$$

such a problem admits a bounded solution. Furthermore,

$$v^1(x) + \chi(x)t^1(n, s) = m(\Xi)\partial_n v^0(\mathcal{O})\mathcal{G}(x), \tag{4.88}$$

where $\mathcal{G}$ is the Poisson kernel, i.e., harmonic function in Ω, which is equal to zero on $\partial\Omega\setminus\mathcal{O}$ and with the singularity $n[\pi(n^2 + s^2)]^{-1}$ at the point $\mathcal{O}$. It is clear that $\mathcal{G} \leq 0$ in Ω.

Theorem 5.1. *The following asymptotic formula holds for the first variation of the energy shape functional with respect to the singular domain perturbation*

$$\mathcal{T}(u^h; \Omega(h)) = \mathcal{T}(v^0; \Omega) + h^2 m(\Xi)\partial_n v^0(\mathcal{O})\partial_n V(\mathcal{O}) + O(h^3) \tag{4.89}$$

where V is given by a solution to the following boundary value problem

$$-\Delta_x V(x) = T'(v^0(x), x), \quad x \in \Omega, \quad V(x) = 0, \quad x \in \Gamma, \tag{4.90}$$

and T' denotes the derivative of integrand in (4.83) with respect to the first variable.

Proof. In the same way as in section §3, see also the comments to the Theorem 4.2, the estimate is obtained

$$\|u^h - V^h; H^1(\Omega(h))\| \leq ch^3.$$

Hence

$$\Big|\int_{\Omega(h)} (T(u^h; x) - T(V^h; x))dx\Big| \leq ch^3. \tag{4.91}$$

Beside that

(4.92)

$$\begin{aligned}
&\Big|\int_{\Omega(h)} T(V^h; x)dx - \int_{\Omega(h)} \Big(T(v^0; x) + T'(v^0; x)\Big(\chi(x)hw^1(\xi) + \chi(x)h^2w^2(\xi) + h^2X_h(x)v^2(x)\Big)\Big)dx\Big| \\
&\qquad \leq ch^2 \int_{\Omega(h)} \Big(\chi(x)^2|w^1(\xi) + hw^2(\xi)|^2 + h^2X_h(x)^2|v^2(x)|^2\Big)dx \\
&\qquad \leq ch^2\Big(\int_0^d \Big(1 + \frac{r}{h}\Big)^2 rdr + h^2\Big)dx \leq ch^4(1 + |\ln h|).
\end{aligned}$$

Observing the relation

$$\begin{aligned}
\int_{\Omega(h)} T(v^0(x), x)dx &= \int_{\Omega} T(v^0(x); x)dx + \int_{\Omega\setminus\Omega(h)} (T(v(\mathcal{O}); \mathcal{O}) + O(r))\,dx \\
&= \int_{\Omega} T(v^0(x), x)dx + O(h^3),
\end{aligned} \tag{4.93}$$

$$h^2\Big|\int_{\Omega(h)} T'(v^0; x)\chi(x)w^2(\xi)dx\Big| \leq ch^2 \int_0^d (1 + \frac{r}{h})^2 rdr \leq ch^4(1 + |\ln h|),$$

we process the integral

$$I(h) = h\int_{\Omega(h)} T'(v^0(x), x)\Big(\chi(x)w^1(\xi) + h\chi_h(x)v^2(x)\Big)dx. \tag{4.94}$$

Since v^2 is a bounded function and the difference $\widetilde{w}^1(\xi) = w^1(\xi) - t^1(\xi)$ (see formulae (4.10), (4.11) and (4.87)) decays at the rate $O(\rho^{-2})$, the same argument as already used in

(4.93), with the exchanges and limit passages $w^1 \to t^1, \chi_h \to 1$ and $\Omega(h) \to \Omega$ leads to the precision $O(h^3)$ in (4.94). Therefore, in view of representation (4.88) we find that

$$I(h) = h^2 m(\Xi)\partial_n v^0(O) \int_{\Omega} T'(v^0; x)\mathcal{G}(x)dx + O(h^3). \tag{4.95}$$

By the definition of the Poisson kernel, which integrates the equation in problem (4.90) with a given right-hand side, determines the normal derivative of the solution at the point O. Thus, the relations (4.91)-(4.95) lead to (4.89) ■

Corollary 5.1. *For the potential energy*

$$\Pi(u^h; \Omega(h)) = \frac{1}{2}\int_{\Omega(h)} |\nabla_x u^h(x)|^2 dx - \int_{\Omega(h)} f(x)u^h(x)dx \tag{4.96}$$

the Green formula and Theorem 5.1 give as a result the asymptotic decomposition

$$\Pi(u^h; \Omega(h)) = -\frac{1}{2}\int_{\Omega(h)} f(x)u^h(x)dx = \Pi(v^0; \Omega) - \frac{h^2}{2} m(\Xi)|\partial_n v^0(O)|^2 + O(h^3) \tag{4.97}$$

since for $T(u^h(x), x) = f(x)u^h(x)$ problems (4.90) and (4.85) coincide.

Let us note that, by the inclusion $\Omega(h) \subset \Omega$, the functional (4.96) is minimised on the smaller class $\mathring{H}^1(\Omega(h))$ compared to the class $\mathring{H}^1(\Omega)$ for the functional

$$\Pi(v^0; \Omega) = \frac{1}{2}\int_{\Omega} |\nabla_x v^0(x)|^2 dx - \int_{\Omega} f(x)v^0(x)dx,$$

which means that we have the inequality $\Pi(u^h; \Omega(h)) \geq \Pi(v^0; \Omega)$. The latter inequality is in accordance with the inequality $m(\Xi) < 0$ given in Lemma 4.1.■

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